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1. Introduction

In a previous paper (Amaral 2007) we introduced several issues related to the definition of convexity in semimetric spaces that is, spaces *(E,d)* such that *E* is a fundamental set and *d* is a real function $d: E \times E \rightarrow R$ that satisfies all the conditions of a metric excluding the triangular inequality. In that paper we explicitly postponed to a future paper the discussion on the possibilities of defining a significant topology for a semimetric space. The aim of the present paper is to discuss this issue. In section 1 we mention already known results or results easily derived from known ones. In section 2 we used these results to obtain a sufficient condition to define a topology on a semimetric space.

1 Pseudo-open sets and structurally continuous spaces

Let *(E,d)* be a semimetric space. Let *N(x,a)* be the open ball with centre at *x* and radius *a*. We define

Definition *(Pseudo-open set).* *A non-empty set* $A \subset E$ *is a pseudo-open set if and only if* $A = \cup_{x,a} N(x,a)$ *for all the x of A and all the a such that for each x,* $N(x,a) \subset A$

Compare the definition with the usual definition of open set:

A non-empty set *A* of the space *(E,d)* is an open set if and only if for each element *x* of *A* there exists $a > 0$ such that $N(x,a) \subset A$.

It is easy to see that if a semimetric space is a metric space a non-empty set *A* is open if and only it is a pseudo-open set. However this is not necessarily the case for non-metric semimetric spaces. Surely, in any semimetric space any non-empty open set is a pseudo-open set but the converse is not necessarily true.

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However we can find sufficient conditions such that for a given semimetric space satisfying those conditions a pseudo-open set is an open set.

Let us begin with the following definition,

Definition ***(Structurally continuous semimetric space).*** *A semimetric space (E,d) is structurally continuous if and only if for any a and b of E and p of R, $p > 0$, there is a $N(b,q)$ of b such that $|d(a,x) - d(a,b)| < p$ for all the x of $N(b,q)$.*

Remark 1. The concept of structurally continuous space is related mainly to the continuity of the function d. It is not necessarily a space such that for each $r > 0$ there is a y distinct from x such that $y \in N(x,r)$. Actually, a space for which there is a $p^* > 0$ such that $N(x,p) = \{x\}$ for all the x of E and all the real numbers p, $p < p^*$, is structurally continuous.

Theorem 1 *Any metric space is structurally continuous*.

Proof. Let a, b and x be elements of E .Using the triangular inequality we have

$$|d(a,x) - d(a,b)| \leq d(x,b).$$

Therefore if we choose a $N(b,q)$ with $q < p$ we obtain the result. □

Structurally continuous non-metric spaces can sometimes be obtained from metric spaces in an easy way. For example, if d^* is a metric and if d is a function such that $d(x,y) = f(d^*(x,y))$ with f a real, strictly increasing function such that $f(z) = 0$ if and only if $z=0$, satisfying the Holder condition $|f(u) - f(v)| \leq M|u-v|^n$ with $M > 0$ and $n>0$, then d is a semimetric and (E,d) is a structurally continuous semimetric space.

The importance of the concept is illustrated by the following theorem:

Theorem 2 *For any structurally continuous, semimetric space a non-empty set A is an open set if and only if A is a pseudo-open set*.

We have only to prove that each pseudo-open set is open.

Proof. By definition $A = \cup_{x,a} N(x,a)$. Let z be an element of A. Then z belongs to a ball $N(x,a^*)$. Since (E,d) is structurally continuous, for every x and z of E and $p>0$ there exists $N(z,q(p))$ of z such that $|d(x,w) - d(x,z)| < p$ for every w belonging to $N(z,q)$. Since $d(x,z)$

$= a^*- h$ with $h > 0$, we can choose a value for p, p^*, such that $p^* < h$. Let w* be any one of those w belonging to $N(z,q(p^*))$.

Due to structural continuity

$d(x,w^*) < d(x,z) + p^*$

$d(x,w^*) < a^*- h + p^* < a^*$

so that w^* belongs to $N(x,a^*)$ and $N(z,q(p^*)) \subset N(x,a^*) \subset A$. □

Corollary 1. *Any open ball of a structurally continuous semimetric space is an open set*.

This means that in such a space we can choose as a base of topology of open sets at a point the family of all the open balls. Open balls are also considered as neighbourhoods of the respective centres.

Corollary 2. *Every structurally continuous semimetric space is a Hausdorff space. (A Hausdorff space is space such that for two distinct points x and y in the space there are two open sets A and B such that* $x \in A$, $y \in B$ *and* $A \cap B = \emptyset$*).*

Proof. Since x and y are distinct $d(x,y) > 0$. Choose a $p < d(x,y)/2$.

As the space is structurally continuous there is a $N(y,q)$ with $q=q(p)$ such that

$|d(x,z) - d(x,y)| < d(x,y)/2$ for all the z of $N(y,q)$.

That is

$d(x,z) > d(x,y) – d(x,y)/2 = d(x,y)/2 > p$ so that z does not belong to $N(x,p)$. Therefore

$N(y,q) \cap N(x,p) = \emptyset$.

Since by Corollary 1 open balls are open sets we obtain the intended result. □

Coollary 3. *Every non-empty set* $\phi(x, y)$ *of a structurally continuous semimetric space* (E,d) *is an open set.*

(**Observation**. *As in Amaral (2017) by the symbol* $\phi(x, y)$*, for any x and y of E we denote the set of all elements z of E such that* $d(x,y) > d(x,z) + d(z,y)$*. Of course for a semimetric space that is a metric space all those sets are empty).*

Proof. Consider a z belonging to $\phi(x, y)$. We have

$d(x,y) - d(x,z) - d(z,y) \equiv H > 0$ Choose two positive numbers, m, n such that $m+n < H$.

Obviously

$d(x,y) = H + d(x,z) + d(z,y) > m+n + d(x,z) + d(z,y)$

Since (E,d) is structurally continuous we have

$d(x,w) < d(x,z) + m$

$d(y,v) < d(y,z) + n$

respectively for all the w of a given ball $N(z,r)$ and all the v of a given ball $N(z,s)$.

Therefore for all the u of the ball $N(z,r^*)$ where $r^* \equiv min(r,s)$ we have

$d(x,u) < d(x,z) + m$

$d(y,u) < d(y,z) + n$

so that

$d(x,u) + d(u,y) < m+n + d(x,z) + d(z,y) < d(x,y)$

and u belongs to $\phi(x, y)$.

Since this happens for all the u of $N(z,r^*)$ we have $N(z,r^*) \subset \phi(x, y)$ and $\phi(x, y)$ is open. □

As the empty set is considered open by definition we have the following version of the corollary

Corollary 3*. *For all structurally continuous semimetric (metric or non-metric) spaces every $\phi(x, y)$ set is open.*

Since the union of the sets of any family of open sets is an open set we have

Corollary 4. *Let Z be the set of all the elements z of E such that there are two elements x, y of E, $x \neq y \neq z$, such that $z \in \phi(x, y)$. Then Z is an open set.*

2. Sufficient condition for structural continuity

With the following theorem we provide a sufficient condition for a semimetric space to be structurally continuous.

Theorem 3. *If (E,d) is such that for every x and y of E,* $inf\{d(y,u_\lambda)\} > \varepsilon(y) > \varepsilon > 0$*, where the* u_λ *are the elements of E that determine all the sets* $\phi(x, u_\lambda)$ *to which y belongs then E is structurally continuous*.

Proof. First note that by the definition of sets $\phi(x,u_\lambda)$ if y belongs to the set , y is not identical to u_λ, so that the condition $inf\{d(y,u_\lambda)\} > \varepsilon(y) > \varepsilon > 0$ makes sense.

Suppose that y belongs to Z (Corollary 4 above). For any $p > 0$ and x, y of E choose ε^* such that $\varepsilon^* < min\ (\varepsilon,p)$ and a w of E such that $d(y,w) < \varepsilon^*$. If there is no w, $w \neq y$ belonging to $N(y, \varepsilon^*)$, the proof is still valid (see **Remark 1** above). Therefore we can say that y does not belong to $\phi(x, w)$.

The same considerations for w instead of y allow us to say that w does not belong to

$\phi(x, y)$, so that we have

$$d(x,w) \leq d(x,y) + d(y,w)$$

$$d(x,y) \leq d(x,w) + d(w,y)$$

Therefore

$$\left| d(x,w) - d(x,y) \right| \leq d(y,w) < \varepsilon^* < p$$

If y does not belong to Z we have for all the w of E

$$d(x,w) \leq d(x,y) + d(y,w)$$

$$\left| d(x,w) - d(x,y) \right| \leq d(y,w)$$

Given a $p > 0$ we choose a $q(p) < p$ so that we have $d(y,w) < p$ and

$$\left| d(x,w) - d(x,y) \right| < p$$

The same for w if it does not belong to Z.□

We obtain easily the following important corollary:

Corollary. *If for every u and v of E, the set* $F \equiv \cup_{u,v} \phi(u,v)$ *has at most a finite number of elements and each x of E belongs at most to a finite number of sets* $\phi(u,v)$ *then* (E, d) *is structurally continuous.*

References

Amaral, João Ferreira do, 2017; *Convexity in semi-metric spaces, decision theory and consumer theory*. REM Working Papers. November .