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REM Working Paper 018-2017

December 2017

REM – Research in Economics and Mathematics
Rua Miguel Lúpi 20,
1249-078 Lisboa,
Portugal

ISSN 2184-108X

Pricing American Call Option by the Black-Scholes Equation with a Nonlinear Volatility Function

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December 3, 2017

Abstract

In this paper we analyze a nonlinear Black-Scholes equation for pricing American style call option in which the volatility may depend on the underlying asset price and the Gamma of the option. We study the generalized Black-Scholes equation by means of transformation of the free boundary problem (variational inequalities) into the so-called Gamma equation for the new variable $H = S\partial_S^2 V$. Moreover, we reformulate our new problem with PSOR method and construct an effective numerical scheme for discretization of the Gamma equation. Finally, we solve numerically our nonlinear complementarity problem applying PSOR method.

*Keywords:*American option pricing, nonlinear Black-Scholes equation, variable transaction costs, PSOR method.

1 Introduction

In the financial market, the price of a European option can be computed from a solution to the well-known Black–Scholes linear parabolic equation derived by Black and Scholes in [5]. A European call option gives its owner the right but not obligation to purchase an underlying asset at the expiration price E at the expiration time T. In this paper, we consider American style options which, as it is known, can be exercised anytime t in the time interval $[0, T]$. The classical linear Black Scholes model was derived under several restrictive assumptions, namely no transaction costs, frictionless, liquid and complete market, etc. However, we need more realistic models in the market data analysis in order to cover the disadvantages of the classical Black-Scholes theory. In this paper, we focus on the model which takes into account non-trivial transaction costs. This leads to the generalized Black-Scholes equation with the nonlinear volatility function $\hat{\sigma}$ which depends

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on the product $H = S\partial_S^2 V$ of the underlying asset price S and the second derivative (Gamma) of the option price V.

$$\partial_t V + \frac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2 \partial_S^2 V + (r-q)S\partial_S V - rV = 0, \tag{1}$$

where $r, q \geq 0$ are the interest rate and the dividend yield, respectively. The price $V(t,S)$ of a call option is then a solution to the nonlinear parabolic equation (1) on the underlying stock $S > 0$ at the time $t \in [0,T]$ subjected to the terminal pay-off diagram

$$V(T,S) = (S-E)^+,$$

where $T > 0$ is the time of maturity and $E > 0$ is the exercise price.

One of the first nonlinear models taking into account transaction costs is the jumping volatility model by Avellaneda, Lévy and Paras [2]. The nonlinearity of the original Black-Scholes model can also arise from the feedback and illiquid market effects due to large traders choosing given stock-trading strategies (Frey and Patie [11], Frey and Stremme [12]), imperfect replication and investors preferences (Barles and Soner [4]), risk from unprotected portfolio (Kratka [19], Jandačka and Ševčovič [18]). In this paper we are mainly concerned with a new nonlinear model derived recently by Ševčovič and Žitňanská [27] for pricing call or put options in the presence of variable transaction costs. This model generalizes the well-known Leland model with constant transaction costs (c.f. [21], [16]) and the Amster et al. model [1] with linearly decreasing transaction costs.

In this paper we study an American call option which price can be computed by means of the generalized Black-Scholes equation with the nonlinear volatility function (1). If the volatility function is constant then it is well known that American options can be priced by means of a solution to a linear complementarity problem (c.f. Kwok [20]). Similarly, for the nonlinear volatility model, one can construct a nonlinear complementarity problem involving the variational inequality for the left-hand side of (1) and the inequality $V(t,S) \geq (S-E)^+$. However, due to the fully nonlinear character of the differential operator in (1), the direct computation of the nonlinear complementarity becomes harder and unstable. Therefore, we reformulate the nonlinear complementarity problem in terms of a new transformed variable for which the differential operator has the form of a quasi-linear parabolic operator. More precisely, for the European style of an option Ševčovič, Jandačka and Žitňanská in [25] and [27] derived a transformation technique (referred to as the Gamma transformation) and showed how the fully nonlinear parabolic equation (1) can be transformed to a quasilinear porous-media type of a parabolic equation

$$\partial_\tau H - \partial_u^2 \beta(H) - \partial_u \beta(H) - (r-q)\partial_u H + qH = 0 \tag{2}$$

for the transformed quantity $H(\tau,u) = S\partial_S^2 V(t,S)$ where $\tau = T - t$, $u = \ln(S/E)$ and

$$\beta(H) = \frac{1}{2}\hat{\sigma}(H)^2 H.$$

In order to apply this transformation for American style options we derive the nonlinear complementarity problem for the transformed variable H and we solve the variational problem by means of the projected successive over relaxation method (c.f. Kwok [20]). Using this method we compute American style call option prices for the nonlinear model that considers variable transaction costs.

The paper is organized as follows. In section 2, we present a nonlinear option pricing model under variable transaction costs. Section 3 is devoted to the transformation of the free boundary problem (variational inequalities) to the so-called Gamma equation. In section 4, we present a reformulation of the problem with PSOR method applying efficient numerical scheme for the Gamma equation based on finite volume method. Finally, in section 5, we show numerical experiments for the option price of the transformed problem.

2 European and American option pricing by the Black-Scholes equation with a nonlinear volatility function

In the original Black-Scholes theory continuous hedging of the portfolio including underlying stocks and options is allowed. In the presence of transaction costs for purchasing and selling the underlying stock, this continuous feature may lead to an infinite number of transaction costs. More precisely, the total transaction costs may become unbounded.

One of the basic nonlinear models including transaction costs is the Leland model [21] for option pricing in which the possibility of rearranging portfolio at discrete time can be relaxed. Recall that, in the derivation of the Leland model [16, 17, 21], it is assumed that the investor follows the delta hedging strategy in which the number δ of bought/sold underlying assets depends on the delta of the option, i.e. $\delta = \partial_S V$. Then, applying self-financing portfolio arguments, one can derive the extended version of the Black Scholes equation

$$\partial_t V + (r-q)S\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V - r_{TC} S = 0. \tag{3}$$

Here the transaction cost measure r_{TC} is given by

$$r_{TC} = \frac{\mathbb{E}[\Delta TC]}{S\Delta t}, \tag{4}$$

where ΔTC is the change in transaction costs during the time interval Δt. If $C \geq 0$ represents a percentage of the cost of the sale and purchase of a share relative to the price S then $\Delta TC = \frac{1}{2}CS|\Delta\delta|$ where $\Delta\delta$ is the number of bought ($\Delta\delta > 0$) or sold ($\Delta\delta < 0$) underlying assets during the time interval Δt. The parameter $C > 0$ measuring transaction costs per unit of the underlying asset can be either constant or it may depend on the number of transaction, i. e. $C = C(|\Delta\delta|)$.

Furthermore, assuming the underlying asset follows the geometric Brownian motion $dS = \mu S dt + \sigma S dW$ it can be shown that $\Delta\delta = \Delta\partial_S V \approx \sigma S \partial_S^2 V \Phi\sqrt{\Delta t}$ where $\Phi \sim N(0,1)$ is normally distributed random variable. Hence

$$r_{TC} = \frac{1}{2}\frac{\mathbb{E}[C(\alpha\,|\Phi|)\alpha\,|\Phi|]}{\Delta t}, \tag{5}$$

where $\alpha := \sigma S|\partial_S^2 V|\sqrt{\Delta t}$ (c.f. [26], [18]).

Next we recall a notion of the mean value modification of the transaction cost function introduced by Ševčovič and Žitňanská in [27].

Definition 1 *[27, Definition 1] Let $C = C(\xi)$, $C : \mathbb{R}_0^+ \to \mathbb{R}$, be a transaction costs function. The integral transformation $\tilde{C} : \mathbb{R}_0^+ \to \mathbb{R}$ of the function C defined as follows:*

$$\tilde{C}(\xi) = \sqrt{\frac{\pi}{2}}\mathbb{E}[C(\xi|\Phi|)|\Phi|] = \int_0^\infty C(\xi x)x\, e^{-x^2/2}dx, \tag{6}$$

is called the mean value modification of the transaction costs function. Here Φ is the random variable with a standardized normal distribution, i.e., $\Phi \sim N(0,1)$.

2.1 Constant Transaction Costs - Leland's model

In the case when the transaction cost measure $C = C_0 > 0$ is constant, then using the fact that $\mathbb{E}[|\Phi|] = \sqrt{\frac{2}{\pi}}$, we can express r_{TC} in (5) as follows:

$$r_{TC} = \frac{1}{2}\sigma^2 S\mathrm{Le}\left|\partial_S^2 V\right|.$$

Here C_0 is the constant parameter and $\mathrm{Le} = \sqrt{\frac{2}{\pi}}\frac{C_0}{\sigma\Delta t} > 0$ is the so-called Leland number. Inserting r_{TC} into (3) we obtain the Leland equation:

$$\partial_t V + (r-q)S\partial_S V + \frac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2\partial_S^2 V - rV = 0, \tag{7}$$

with the diffusion term $\hat{\sigma}(S\partial_S^2 V)^2 = \sigma^2(1 - \mathrm{Le}\,\mathrm{sgn}(\partial_S^2 V)) = \sigma^2(1 - \mathrm{Le}\,\mathrm{sgn}(S\partial_S^2 V))$ given by the Leland model (c.f. [16, 17, 21]).

2.2 Non-increasing Transaction Costs Function

Following Amster et al. [1] we can consider a linear non-increasing transaction costs function:

$$C(\xi) = C_0 - \kappa\xi, \quad \text{where} \quad \xi \geq 0, \tag{8}$$

Here $\kappa \geq 0$ is the rate measuring the change of the transaction costs and C_0 is positive constant parameter. The mean value modification function of the Amster model et al. is as follows:

$$\tilde{C}(\xi) = C_0 - \sqrt{\pi/2}\kappa\xi \quad \text{where} \quad \xi \geq 0, \tag{9}$$

where κ and C_0 are the same as in relation (8).

In the real market $C(\xi)$ has to be non-negative but the function (8) may attain negative values provided the transaction amount $|\Delta\delta| = \beta \geq \sqrt{\frac{2}{\pi}}C_0/\kappa$.

2.3 Piecewise Decreasing Transaction Costs Function

Next we want to propose a more realistic example of non-constant transaction costs function and then their relevant mean value modification $\tilde{C}(\xi)$.

In a stylized financial market the transaction costs function should not reach the negative value. In this part we introduce a realistic example of transaction cost (can be seen in Ševčovič and Žitňanská [27]). The advantage of this linear decreasing function is the excluding of the negative values of such a function.

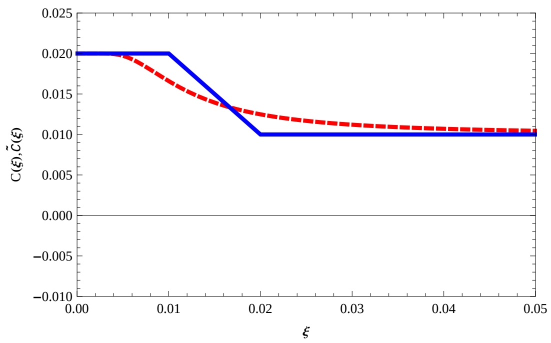

Figure 1: A piecewise linear transaction costs function with $C_0 = 0.02, \kappa = 1, \xi_- = 0.01, \xi_+ = 0.02$ and its mean value modification $\tilde{C}(\xi)$ (dashed line)

Definition 2 *A piecewise linear decreasing transaction costs function is given*

$$C(\xi) = \begin{cases} C_0, & \text{if} \quad 0 \le \xi < \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \text{if} \quad \xi_- \le \xi \le \xi_+, \\ \underline{C}_0, & \text{if} \quad \xi \ge \xi_+. \end{cases} \tag{10}$$

where $\xi_- \le \xi_+$ are given positive constants and as well as κ, C_0 are assumed to be positive. This transaction costs function seems to be more close to reality at which it pays the amount C_0 for the small volume of traded assets, when the traded stocks volume is higher, there is a discount for that and when the trades are very large, it just pays a small constant $\underline{C}_0$. For better understanding, in Fig. 1 we show the graphs of both relevant transaction costs function and its mean value function with the known parameter values.

Proposition 1 *[27, Eq. (24)] Let C_0, κ be the positive constants, then for the piecewise linear function (10) the modified mean value transaction costs function is given by*

$$\tilde{C}(\xi) = C_0 - \kappa \xi \int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} e^{-u^2/2} du, \quad \text{for} \quad \xi \ge 0. \tag{11}$$

Applying integration by parts we can simply deduce the following function (see Žitňanská and Ševčovič [27]).

There is a bound for this mean value transaction costs function $\tilde{C}(\xi)$.

Proposition 2 *[27, Proposition 2.2] Let C_0 be positive in Definition (2). Then the modification transaction costs function in (11) verifies*

$$\underline{C}_0 \le \tilde{C}(\xi) \le C_0 \tag{12}$$

and

$$\lim_{\xi \to \infty} \tilde{C}(\xi) = \lim_{\xi \to \infty} C(\xi) = \underline{C}_0. \tag{13}$$

Proposition 3 *[27, Proposition 2.1] Assume that $C : \mathbb{R}_0^+ \to \mathbb{R}$ is a measurable and bounded function of the transaction costs function as well. Then the price of the option* *based on the variable transaction costs is given by the solution of the following nonlinear Black Scholes PDE*

$$\partial_t V + (r-q)S\partial_S V + \frac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2 \partial_S^2 V - rV = 0, \tag{14}$$

where the nonlinear diffusion coefficient $\hat{\sigma}^2$ *is*

$$\hat{\sigma}(S\partial_S^2 V)^2 = \sigma^2 \left(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma S \left|\partial_S^2 V\right| \sqrt{\Delta t})\frac{\mathrm{sgn}(S\partial_S^2 V)}{\sigma\sqrt{\Delta t}}\right). \tag{15}$$

3 Transformation of the free boundary problem to the Gamma equation

In this section we want to investigate transformation of the free boundary problem (variational inequalities) to the so-called Gamma equation proposed and then developed by Jandačka and Ševčovič [18].

Let us consider the generalized nonlinear Black Scholes equation for European option pricing of the form

$$\partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV = 0, \quad S > 0, t \in (0,T), \tag{16}$$

where

$$\beta(H) = \frac{1}{2}\hat{\sigma}(H)^2 H.$$

Then, making the change of variables $u = \ln(\frac{S}{E})$ and $\tau = T - t$ and computing the second derivative of the equation (16) with respect to u, we derive the so-called Gamma equation, given by

$$\partial_\tau H - \partial_u \beta(H) - \partial_u^2 \beta(H) - (r-q)\partial_u H + qH = 0 \tag{17}$$

More details can be found in Ševčovič and Žitňanská [27].

Lemma 1 *[27, Proposition 3.1, Remark 3.1] Let us consider the Call option with the pay-off diagram* $V(T,S) = (S-E)^+$*. Then the function* $H(\tau,u) = S\partial_S^2 V(t,S)$ *where* $u = \ln(\frac{S}{E})$ *and* $\tau = T - t$ *is a solution to (17) subject to the Dirac initial condition* $H(0,x) = \delta(x)$ *if and only if*

$$V(t,S) = \int_{-\infty}^{+\infty} (S - Ee^u)^+ H(\tau,u)du$$

3.1 American style options

The real advantage of American style contracts over European style contracts is the flexibility that they offer. When you own this type of contracts, it gives you the right to exercise earlier than the expiration date of the contract. More precisely, in mathematical modeling of American options, unlike European style options, there is the possibility of early exercising the contract at some time $t^* \in [0,T)$ prior to the maturity time T. It is fairly saying that the most of the derivative contracts traded in the financial markets are of the American style. As known, an American call option is the contract that gives the right but not the obligation to buy the underlying asset at the strike price E anytime $t \in [0,T]$ prior to the expiration time $t = T$. As in the case of European style options, we are interested in knowing the fair option premium at the starting point $t = 0$ of contracting. In the case of an American call option the challenge is to find the price of the option $V(t,S)$ at the time $t \in [0,T]$ having in view the possible gain if exercising it at that time t. Comparing an American style contract with the European one the relation between the values of these two types of contracts gives an inequality presenting

$$V^A(t,S) \geq V^E(t,S) = (S-E)^+, \quad \forall S \geq 0, \quad t \in [0,T]. \tag{18}$$

For the American call option, if the price of the option $V(t,S)$ at anytime prior to the maturity T is lower than its payoff function $(S-E)^+$ then the policy is to purchase the option and exercise it immediately as we are allowed for these type of contracts. But in this case there would be an arbitrage opportunity for the holder of the option. With respect to the highly demand for trading such an option, the market will increase its price to a value higher or equal to the payoff function and, then, the arbitrage opportunity will be removed. Assuming that American call option on the underlying stock is paying the dividend yield $q \geq 0$, then, for large values of the underlying stock price $S \gg E$, the price of the American call option satisfies

$$V^A(t,S) > V^E(t,S), \quad \text{for each } S > 0,\, t \in [0,T). \tag{19}$$

where $q > 0$ and $r > 0$. It is well-known that pricing an American call option on an underlying stock paying continuous dividend yield $q > 0$ leads to a free boundary problem. In addition to a function $V(t,S)$, we need to find the early exercise boundary function $S_f(t)$ with respect to time $t \in [0,T]$. Furthermore, we note that the function $S_f(t)$ has the following properties:

- If $S_f(t) > S$ for $t \in [0,T]$ then $V(t,S) > (S-E)^+$.
- If $S_f(t) \leq S$ for $t \in [0,T]$ then $V(t,S) = (S-E)^+$.

Remark 1 *Following Kwok [20] (see also [26]) we can also formulate the free boundary problem for pricing the American call option. It consists of finding a function $V(t,S)$ and the early exercise boundary function that solve the Black-Scholes PDE on a time depending domain:*

$$\partial_t V + \frac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2 \partial_S^2 V + (r-q)S\partial_S V - rV = 0, \qquad 0 < S < S_f(t), \tag{20}$$

$$V(T,S) = (S-E)^+, \tag{21}$$

$$V(t,S_f(t)) = S_f(t) - E, \quad \partial_S V(t,S_f(t)) = 1, \quad V(t,0) = 0. \tag{22}$$

3.2 Transformation of the variational inequality

In the presence of transaction costs for buying and selling the underlying stock, we face the nonlinear problem in which we transformed the arising free boundary problem for pricing the American call option into the so-called Gamma equation for the new variable $H = S\partial_S^2 V$.

Lemma 2 *Let $V(t,S)$ be a given function. Assuming that $u=\ln(\frac{S}{E})$, $\tau=T-t$. Define the function $Y(\tau,u)$*

$$Y(\tau,u)=\partial_t V+(r-q)S\partial_S V+S\beta(S\partial_S^2 V)-rV.$$

Then

$$-\partial_\tau H+\partial_u\beta(H)+\partial_u^2\beta(H)+(r-q)\partial_u H-qH=\frac{1}{E}e^{-u}[\partial_u^2 Y-\partial_u Y],$$

where $H(\tau,u)=S\partial_S^2 V(t,S)$

P r o o f. By differentiating the function Y with respect to the variable u and using the fact $\partial_u=S\partial_S$, we obtain

$$\partial_u Y=\partial_t(S\partial_S V)+S(\beta+\partial_u\beta)+(r-q)SH-qS\partial_S V \quad \text{where} \quad S=Ee^u.$$

Furthermore, since

$$\partial_u^2 Y=\partial_t(S\partial_S V+S^2\partial_S^2 V)+(r-q)S(H+\partial_u H)+S(\beta+\partial_u\beta)+S(\partial_x^2\beta+\partial_u\beta)-qS\partial_S V-qH,$$

then

$$\partial_u^2 Y-\partial_u Y=Ee^u\Psi[H], \tag{23}$$

where $\Psi[H]:=-\partial_\tau H+\partial_u\beta(H)+\partial_u^2\beta(H)+(r-q)\partial_u H-qH.$

Remark 2 *For the particular case $Y=0$, we conclude that the function $V(t,S)$ is a solution to the European style option satisfying the nonlinear Black-Scholes equation (14) if and only if the function $H(\tau,u)$ is a solution to the so-called Gamma equation*

$$-\partial_\tau H+\partial_u\beta(H)+\partial_u^2\beta(H)+(r-q)\partial_u H-qH=0.$$

(c.f. [25], [27]).

Lemma 3 *Assuming that*

$$lim_{u\to-\infty}Y(\tau,u)=0 \quad and \quad lim_{u\to-\infty}e^{-u}\partial_u Y(\tau,u)=0,$$

then by applying equation (23) we have

$$\int_{-\infty}^{+\infty}(S-Ee^u)^+\Psi[H(\tau,u)]dx=Y(\tau,u)|_{u=\ln(S/E)}$$

P r o o f.

$$\begin{aligned}
\int_{-\infty}^{+\infty}(S-Ee^u)^+\frac{1}{E}e^{-u}[\partial_u^2 Y-\partial_u Y]du &= \frac{1}{E}\int_{-\infty}^{\ln(S/E)}(Se^{-u}-E)[\partial_u^2 Y-\partial_u Y]du\\
&= \frac{1}{E}\int_{-\infty}^{\ln(S/E)}\left[Se^{-u}\partial_u Y-(Se^{-u}-E)\partial_u Y du\right]\\
&+\underbrace{[(Se^{-u}-E)\partial_u Y]_{-\infty}^{\ln(S/E)}}_{0}\\
&= \frac{1}{E}\int_{-\infty}^{+\infty}E\partial_u Y du=Y(\tau,u)|_{u=\ln(S/E)}\\
&= \partial_t V+(r-q)S\partial_S V+S\beta(S\partial_S^2 V)-rV.
\end{aligned}$$

Theorem 1 *The function $V(t,S)$ is a solution to the nonlinear complementarity problem (NLCP):*

$$V(t,S) \geq (S-E)^+ \quad \textit{and} \quad \partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV \leq 0$$

if and only if for each $S \geq 0$ and $\tau \in [0,T]$ the following inequalities hold:

$$\int_{-\infty}^{+\infty} (S - Ee^u)^+ [H(\tau,u) - H(0,u)] du \geq 0,$$

and

$$\int_{-\infty}^{+\infty} (S - Ee^u)^+ \Psi[H] du \leq 0,$$

where $H(\tau,u) = S\partial_S^2 V(t,S)$ and $V(T,S) = (S-E)^+ = \int_{-\infty}^{+\infty} (S-Ee^u)^+ H(0,u)du$.
P r o o f. It can simply be proved by applying Lemma 2 and Lemma 3.

Remark 3 *For calculating $V(T,S)$ in Theorem 1 we use the fact*

$$H(0,u) = \bar{H}(u), \quad u \in \mathbb{R},$$

where $\bar{H}(u) := \delta(u)$ is the Dirac delta function such that

$$\int_{-\infty}^{+\infty} \delta(u) du = 1, \quad \int_{-\infty}^{+\infty} \delta(u-u_0)\phi(u)du = \phi(u_0),$$

for any continuous function ϕ.

4 Reformulation of the problem with PSOR method

By using the result form contained in Theorem 1, the American call option problem can be rewritten in terms of the function $H(\tau,u)$ as follows:

$$\int_{-\infty}^{+\infty} (S - Ee^u)^+ \left[\partial_\tau H - (r-q)\partial_u H - \partial_u \beta(H) - \partial_u^2 \beta(H) + qH\right] du \geq 0, \quad (24)$$

$$\int_{-\infty}^{+\infty} (S - Ee^u)^+ H(\tau,u) \geq g(S) = (S-E)^+. \quad (25)$$

for any $S \geq 0$ and $\tau \in [0,T]$.

In order to apply the Projected Successive Over Relaxation method (PSOR) (c.f. Kwok [20]) to the inequalities (24)–(25), we need first to discretize the operator

$$-\Psi[H] = \partial_\tau H - (r-q)\partial_u H - \partial_u \beta(H) - \partial_u^2 \beta(H) + qH. \quad (26)$$

In the next, we follow the paper by Ševčovič and Žitňanská [27] in order to derive an efficient numerical scheme for solving the Gamma equation in presence of a general function $\beta(H)$ with the model including variable transaction costs.

4.1 Numerical scheme for the Gamma equation

The proposed numerical discretization is based on the finite volume method. Assume that the spatial interval belongs to $u \in (-L, L)$ for sufficiently large $L > 0$ where the time interval $[0, T]$ is uniformly divided with a time step $k = \frac{T}{m}$ into discrete points $\tau_j = jk$ for $j = 1, 2, \cdots, m$. Furthermore, we divide the spatial interval $[-L, L]$ into a uniform mesh of discrete points $u_i = ih$ where $i = -n, \cdots, n$ with a spatial step $h = \frac{L}{n}$.

This leads to a tridiagonal system for the vector $H^j = (H^j_{-n+1}, \cdots, H^j_{n-1}) \in \mathbb{R}^{2n-1}$. It means that the vector $\Psi[H]^j$ at the time level τ_j is given by $\Psi[H]^j = -(A^j H^j - d^j)$ where the $(2n-1) \times (2n-1)$ matrix A^j has the form

$$A^j = \begin{pmatrix} b^j_{-n+1} & c^j_{-n+1} & 0 & \cdots & 0 \\ a^j_{-n+2} & b^j_{-n+2} & c^j_{-n+2} & & \vdots \\ 0 & . & . & . & 0 \\ \vdots & \cdots & a^j_{n-2} & b^j_{n-2} & c^j_{n-2} \\ 0 & \cdots & 0 & a^j_{n-1} & b^j_{n-1} \end{pmatrix} \tag{27}$$

and the coefficients of the matrix are

$$\begin{aligned} a^j_i &= -\frac{k}{h^2}\beta'_H(H^{j-1}_{i-1}) + \frac{k}{2h}(r-q), \\ c^j_i &= -\frac{k}{h^2}\beta'_H(H^{j-1}_{i}) - \frac{k}{2h}(r-q), \\ b^j_i &= (1+kq) - (a^j_i + c^j_i), \end{aligned}$$

where

$$d^j_i = H^{j-1}_i + \frac{k}{h}\left(\beta(H^{j-1}_i) - \beta(H^{j-1}_{i-1})\right).$$

Finally, with respect to the inequality (25), by means of an integration scheme, the price of the call option can be presented as follows:

$$V(S, T-\tau_j) = h\sum_{i=-n}^{n}(S - Ee^{u_i})^+ H^j_i, \quad j = 1, 2, \cdots, m. \tag{28}$$

Then, the full space-time discretized version of the problem (24)–(25) is given by

$$h\sum_{i=-n}^{n}(S - Ee^{u_i})^+ \left[(A^j H^j)_i - d^j_i\right] \;\geq\; 0, \tag{29}$$

$$h\sum_{i=-n}^{n}(S - Ee^{u_i})^+ H^j_i \;\geq\; g(S) \equiv (S-E)^+. \tag{30}$$

Let us assume that

$$P_{li} \;=\; h[\max(S_l - Ee^{u_i}, 0)] = h[\max(Ee^{v_l} - Ee^{u_i}, 0)] \tag{31}$$

where

$$v_l = \frac{u_{l+1} + u_{l-1}}{2}, \quad \text{for } l = -n, \cdots, n.$$

Remark 4 *The matrix $P = (P_{li})$ is invertible.*

4.2 Applying the PSOR method

In this section we want to solve the problem (29)–(30) making use of the PSOR method. Then, according to (31), we can rewrite (29)–(30) for the American call option in this form

$$\begin{aligned} PAH &\geq Pd \\ PH &\geq g \\ (PAH - Pd)_i(PH - g)_i &= 0 \quad \forall i, \end{aligned}$$

where $A = A^j$, $g_i = (S_i - E)^+$ and $H = H^j$.

This NLCP can be solved by the PSOR algorithm, given by the following iterative scheme:

1. for $k = 0$ set $v^{j,k} = v^{j-1}$,
2. until $k \leq k_{max}$ repeat:

$$\begin{aligned} t_i^{j,k+1} &= \frac{1}{\tilde{A}_{ii}^j}\left(-\sum_{l<i}\tilde{A}_{il}^j v_l^{j,k+1} - \sum_{l>i}\tilde{A}_{il}^j v_l^{j,k} + \tilde{A}_{ii}d_i^j\right), \\ v_i^{j,k+1} &= \max\left\{v_i^{j,k} + \omega(t_i^{j,k+1} - v_i^{j,k}),\, g_i\right\}, \end{aligned}$$

3. set $v^j = v^{j,k+1}$,

where $v^j = PH^j$ for $i = -n, \cdots, n$ and $j = 1, 2, \cdots, m$ and $\tilde{A} = PAP^{-1}$. Here $\omega \in [1, 2]$ is a relaxation parameter which can be tuned in order to speed up convergence process.

Finally, using the value $H^j = P^{-1}v^j$ and equation (28), we can evaluate the price of the option.

5 Numerical experiments

In this section, we focus our attention on numerical experiments for computing an American style call option price based on the nonlinear Black-Scholes equation that includes a piecewise linear decreasing transaction costs function. In Fig. 2, we show the corresponding function $\beta(H)$ given by

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\mathrm{sgn}(H)}{\sigma\sqrt{\Delta t}}\right)H.$$

Here $\tilde{C}$ is the modified transaction costs function.

The model parameters corresponds to the nonlinear variable transaction cost model are given in Table 1, where Δt is the time interval between two consecutive portfolio rearrangements, the maturity time T, the historical volatility σ, the dividend yield q, the strike price E and r is the risk free interest rate.

For the given numerical parameters in Table 1, we present option values V_{vtc} for the underlying asset prices calculated by numerical solutions for both Bid and Ask option

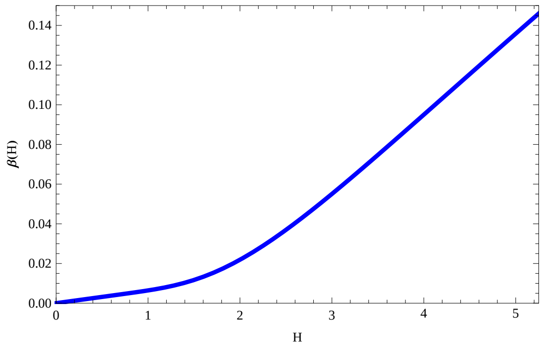

Figure 2: A graph of the function $\beta(H)$ related to the piecewise linear decreasing transaction costs function (see [18]).

Model params	Numerical params
$C_0 = 0.02$	T=1
$\kappa = 0.3$	E=50
$\xi_- = 0.05$	m=200, 800
$\xi_+ = 0.1$	n=250, 500
$\Delta t = 1/261$	h=0.01
$\sigma = 0.3$	$\tau^* = 0.005$
r=0.011	$k = T/m$
q= 0.008	L=2.5

Table 1: Model and numerical parameter values for calculation of numerical experiments.

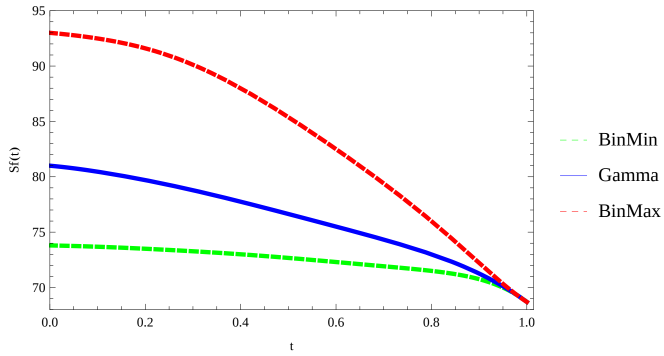

Figure 3: The early exercise boundary function $S_f(t), t \in [0,T]$, computed for the model with variable transaction costs (dashed line Gamma) and comparison with early exercise boundary computed by means of binomial trees with constant volatilities σ_{min} (bottom curve) and σ_{max} (top curve).

prices in Table 2. For the Bid price, the lower line is related to the solution of the binomial tree method with the lower volatility $\hat{\sigma}^2_{min} = \sigma^2(1 - C_0\sqrt{\frac{2}{\pi}\frac{1}{\sigma\sqrt{\Delta t}}})$, whereas the upper line is related to the solution with a higher volatility $\hat{\sigma}^2_{max} = \sigma^2(1 - \underline{C_0}\sqrt{\frac{2}{\pi}\frac{1}{\sigma\sqrt{\Delta t}}})$. As well as for the Ask price the lower line corresponds to the solution of the binomial tree method with the lower volatility $\hat{\sigma}^2_{min} = \sigma^2(1 + \underline{C_0}\sqrt{\frac{2}{\pi}\frac{1}{\sigma\sqrt{\Delta t}}})$, whereas the upper line corresponds to the solution with a higher volatility $\hat{\sigma}^2_{max} = \sigma^2(1 + C_0\sqrt{\frac{2}{\pi}\frac{1}{\sigma\sqrt{\Delta t}}})$.

Remark 5 *In the case of a European style option, it can be shown analytically by using the parabolic comparison principle that*

$$V_{\sigma_{min}}(S,t) \leq V_{vtc}(t,S) \leq V_{\sigma_{max}}(t,S), \quad \forall S > 0, t \in [0,T].$$

For more details we refer to [27]. For the case of American style options, these inequalities can be observed in Table 2.

In Table 3, we present a comparison of results achieved by our method based on the Gamma equation in the special case of constant transaction costs and obtained by well-known method based on binomial trees (with the number of nodes equal to 100 and 200), whereas $C_0 = 0.02, \kappa_0 = 0.3, \xi_- = 0.05, \xi_+ = 0.1$ and $\underline{C_0} \leftarrow C_0 - \kappa_0(\xi_+ - \xi_-)$.

In Fig. 4 we plot the graphs of the solutions $V_{vtc}(t,S)$ for both bid and ask price with the lower volatility σ_{min} and the higher volatility σ_{max}, respectively.

Finally, in Fig. 3 we present the free boundary function $S_f(t)$ obtained by our method with variable transaction costs function for bid option value compared to the binomial trees with $\sigma_{min}, \sigma_{max}$ in which parameter values are given by $E = 50, \sigma = 0.3, r = 0.011, q = 0.008, T = 1$.

	$n = 250, m = 200$				$n = 500, m = 800$		
S	V_{BinMin}	$V_{Bid_{vtc}}$	V_{BinMax}	S	V_{BinMin}	$V_{Bid_{vtc}}$	V_{BinMax}
40	0.0320	0.0513	1.3405	40	1.4511	1.6594	2.8670
42	0.1075	0.3252	1.8846	42	2.0137	2.3869	3.6039
44	0.2901	0.8232	2.5527	44	2.6979	3.2309	4.4371
46	0.6535	1.5097	3.3483	46	3.5064	4.1868	5.3645
48	1.2675	2.3859	4.2711	48	4.4382	5.2488	6.3833
50	2.1740	3.4244	5.3175	50	5.4897	6.4133	7.4889
52	3.3738	4.6126	6.4817	52	6.6553	7.6764	8.6772
54	4.8304	5.9521	7.7555	54	7.9270	9.0342	9.9423
56	6.4862	7.4377	9.1295	56	9.2959	10.4824	11.2798
58	8.2809	9.0643	10.5943	58	10.7532	12.0179	12.6832
60	10.1635	10.8273	12.1397	60	12.2892	13.6385	14.1481

	$n = 250, m = 200$				$n = 500, m = 800$		
S	V_{BinMin}	$V_{Ask_{vtc}}$	V_{BinMax}	S	V_{BinMin}	$V_{Ask_{vtc}}$	V_{BinMax}
40	1.4511	1.6594	2.8670	40	1.4420	1.6692	2.8519
42	2.0137	2.3869	3.6039	42	2.0027	2.3945	3.5870
44	2.6979	3.2309	4.4371	44	2.6851	3.2412	4.4187
46	3.5064	4.1868	5.3645	46	3.4922	4.2134	5.3450
48	4.4382	5.2488	6.3833	48	4.4231	5.2601	6.3627
50	5.4897	6.4133	7.4889	50	5.4742	6.4300	7.4678
52	6.6553	7.6764	8.6772	52	6.6395	7.6922	8.6557
54	7.9270	9.0342	9.9423	54	7.9115	9.2167	9.9211
56	9.2959	10.4824	11.2798	56	9.2812	11.0264	11.2586
58	10.7532	12.0179	12.6832	58	10.7393	12.2017	12.6628
60	12.2892	13.6385	14.1481	60	12.2763	13.6505	14.1283

Table 2: Bid (top table) and Ask (bottom table) American Call option values $V_{Bid_{vtc}}$ and $V_{Ask_{vtc}}$ obtained from the numerical solution of the nonlinear model with variable transaction costs for different meshes. Comparison to the option prices V_{BinMin} and V_{BinMax} computed by means of binomial trees for constant volatilities σ_{min} and σ_{max}.

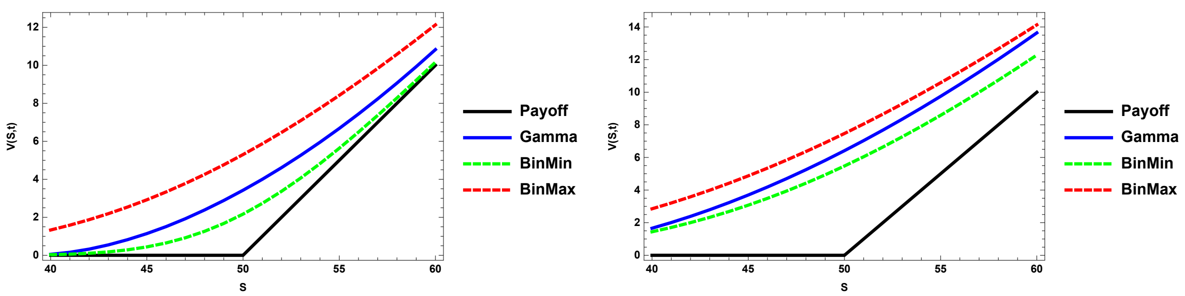

Figure 4: The American Bid (left) and Ask (right) call option price $V(t, S)$ with $n = 500, m = 800$ calculated by means of the model with variable transaction costs in comparison to solutions $V_{\sigma_{min}}, V_{\sigma_{max}}$ calculated by the binomial trees with constant volatilities σ_{min} and σ_{max}.

	$n = 250, m = 200$					$n = 500, m = 800$			
S	$V_{Ask_{vtc}}$	V_{BinMin}	$V_{Ask_{vtc}}$	V_{BinMin}	S	$V_{Ask_{vtc}}$	V_{BinMax}	$V_{Ask_{vtc}}$	V_{BinMax}
40	1.4737	1.4511	1.4634	1.4420	40	2.8827	2.8670	2.8663	2.8519
42	2.2417	2.0137	2.110	2.002	42	3.6273	3.6039	3.5923	3.5870
44	2.7156	2.6979	2.7025	2.6851	44	4.4618	4.4371	4.4067	4.4187
46	3.5287	3.5064	3.5193	3.4922	46	5.3945	5.3645	5.3561	5.3450
48	4.4572	4.4382	4.4498	4.4231	48	6.4095	6.3833	6.3515	6.3627
50	5.5019	5.4897	5.4996	5.4742	50	7.5002	7.4889	7.4710	7.4678
52	6.6993	6.6553	6.6684	6.6395	52	8.7049	8.6772	8.6682	8.6557
54	7.9537	7.9270	7.9350	7.9115	54	9.9765	9.9423	9.9326	9.9211
56	9.3367	9.2959	9.3145	9.2812	56	11.3071	11.2798	11.2742	11.2586
58	10.8015	10.7532	10.7683	10.7393	58	12.7103	12.6832	12.6790	12.6628
60	12.3369	12.2892	12.3189	12.2763	60	14.1640	14.1481	14.1374	14.1283

Table 3: Ask call option values of the numerical solution of the model under constant volatility in comparison to the Binomial tree method (with $n = 100$ and $n = 200$, respectively).

6 Conclusions

In this paper we have investigated a nonlinear generalization of the Black-Scholes equation for pricing American style call options assuming variable transaction costs for trading the underlying assets. In this way, we presented a model that addresses a more realistic financial framework than the classical Black-Scholes model. From the mathematical point of view, we studied a problem that consists of a fully nonlinear parabolic equation in which the nonlinear diffusion coefficient may depend on the second derivative of the option price. Furthermore, for the American call option we have transformed the nonlinear complementarity problem into the so called Gamma equation. We have reformulated our new problem using PSOR method and presented an effective numerical scheme for discretizing the Gamma equation. Then, we made some numerical computations for the model with variable transaction costs and exhibited a comparison between the respective early exercise boundary function and the early exercise boundary computed by means of binomial trees with constant volatilities.

Acknowledgements:

This research was supported by the European Union in the FP7-PEOPLE-2012-ITN project STRIKE - Novel Methods in Computational Finance (304617), the project CEMAPRE MULTI/00491 financed by FCT/MEC through national funds and the Slovak research Agency Project VEGA 1/0780/15.

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