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Pricing Perpetual Put Options by the Black-Scholes Equation with a Nonlinear Volatility Function

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Abstract

We investigate qualitative and quantitative behavior of a solution to the problem of pricing American style of perpetual put options. We assume the option price is a solution to a stationary generalized Black-Scholes equation in which the volatility may depend on the second derivative of the option price itself. We prove existence and uniqueness of a solution to the free boundary problem. We derive a single implicit equation for the free boundary position and the closed form formula for the option price. It is a generalization of the well-known explicit closed form solution derived by Merton for the case of a constant volatility. We also present results of numerical computations of the free boundary position, option price and their dependence on model parameters.

Keywords: Option pricing, nonlinear Black-Scholes equation, perpetual American put option, early exercise boundary

2000 MSC: 35R35, 91B28, 62P05

1 Introduction

In a stylized financial market, the price of a European option can be computed from a solution to the well-known Black–Scholes linear parabolic equation derived by Black and Scholes in [5], and, independently by Merton (cf. Kwok [18], Dewynne *et al.* [7], Hull [15]). A European call (put) option is the right but not obligation to purchase (sell) an underlying asset at the expiration price E at the expiration time T.

In contrast to European options, American style of options can be exercised anytime in the temporal interval $[0, T]$ with the specified time of obligatory expiration at $t = T$. A mathematical model for pricing American put options leads to a free boundary problem consisting in construction of a function $V = V(S, t)$ together with the early exercise boundary profile $S_f : [0, T] \to \mathbb{R}$ satisfying the following conditions:

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1. V is a solution to the Black–Scholes partial differential equation:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + rS\partial_S V - rV = 0 \tag{1}$$

   defined on the time dependent domain $S > S_f(t)$ where $0 < t < T$. Here σ is the volatility of the underlying asset price process, $r > 0$ is the interest rate of a zero-coupon bond. A solution $V = V(S,t)$ represents the price of an option if the price of the underlying asset is $S > 0$ at the time $t \in [0,T]$;

2. V satisfies the terminal pay-off condition:

$$V(S,T) = \max(E - S,\, 0); \tag{2}$$

3. and boundary conditions for the put option:

$$V(S_f(t),t) = E - S_f(t), \quad \partial_S V(S_f(t),t) = -1, \quad V(+\infty,t) = 0, \tag{3}$$

   for $S = S_f(t)$ and $S = \infty$. Here $(x)_+ = \max(x,0)$ denotes the positive part of x.

If the diffusion coefficient $\sigma > 0$ in (1) is constant then (1) is a classical linear Black–Scholes parabolic equation derived by Black and Scholes in [5]. If we assume the volatility coefficient $\sigma > 0$ is a function of the solution V then equation (1) with such a diffusion coefficient represents a nonlinear generalization of the Black–Scholes equation. We recall that the American style of the put option has been investigated by many authors (c.f. Kwok [18] and references therein). Accurate analytic approximations of the free boundary position have been derived Stamicar, Ševčovič and Chadam [22], Evans, Kuske and Keller [8], and by S. P. Zhu and Lauko and Ševčovič in recent papers [26] and [19] dealing with analytic approximations on the whole time interval.

In this paper we focus our attention to the case when the diffusion coefficient σ^2 may depend on the asset price S and the second derivative $\partial_S^2 V$ of the option price. More precisely, we will assume that

$$\sigma = \sigma(S\partial_S^2 V)\,, \tag{4}$$

i.e. σ depends on the product $S\partial_S^2 V$ of the asset price S and the second derivative (Gamma) of the option price V. Recall that the nonlinear Black–Scholes equation (1) with the volatility σ having the form of (4) arises from option pricing models taking into account nontrivial transaction costs, market feedbacks and/or risk from a volatile (unprotected) portfolio. The linear Black–Scholes equation with constant σ has been derived under several restrictive assumptions like e.g., frictionless, liquid and complete markets, etc. Such assumptions have been relaxed in order to model the presence of transaction costs (see e.g., Leland [20], Hoggard *et al.* [14], Avellaneda and Paras [2]), feedback and illiquid market effects due to large traders choosing given stock-trading strategies (Frey [9], Frey and Patie [10], Frey and Stremme [11], Schönbucher and Wilmott [21]), imperfect replication and investor's preferences (Barles and Soner [4]), risk from unprotected portfolio (Kratka [17], Jandačka and Ševčovič [16] or [23]).

In the Leland model (generalized for more complex option strategies by Hoggard *et al.*) the volatility is given by $\sigma^2 = \sigma_0^2(1 + \mathrm{Le}\,\mathrm{sgn}(\partial_S^2 V))$ where $\sigma_0 > 0$ is the constant historical volatility of the underlying asset price process and $\mathrm{Le} > 0$ is the so-called Leland number.

Another nonlinear Black–Scholes model has been derived by Frey *et al.* (see [9, 11, 10]). In this model the asset dynamics takes into account the presence of feedback effects due to a large trader choosing his/her stock-trading strategy (see also [21]). The diffusion coefficient σ is again non-constant:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 \left(1 - \mu S\partial_S^2 V\right)^{-2}, \tag{5}$$

where $\sigma_0^2, \mu > 0$ are constants. Recently, explicit solutions to the Black–Scholes equation with varying volatility (5) have been derived by Bordag and Chankova [6].

Our last example of the Black–Scholes equation with a non-constant volatility is the so-called Risk Adjusted Pricing Methodology model proposed by Kratka in [17] and revisited by Jandačka and Ševčovič in [16]. In the Risk adjusted pricing methodology model (RAPM) the purpose is to optimize the time-lag between consecutive portfolio adjustments in such way that the sum of the rate of transaction costs and the rate of a risk from unprotected portfolio is minimal. In this model, the volatility is again non-constant:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 \left(1 + \mu (S\partial_S^2 V)^{\frac{1}{3}}\right). \tag{6}$$

By $\sigma_0 > 0$ we denoted the constant historical volatility of the asset price returns and $\mu = 3(C^2 R/2\pi)^{\frac{1}{3}}$, where $C, R \geq 0$ are nonnegative constants representing the transaction cost measure and the risk premium measure, respectively. (see [16] for details).

Another important contribution in this direction has been presented in the paper [1] by Amster, Averbuj, Mariani and Rial, where the transaction costs are assumed to be a non-increasing linear function of the form $C(\xi) = C_0 - \kappa\xi$, (C_0, $\kappa > 0$), depending on the volume of traded stocks $\xi \geq 0$ that is needed to hedge the replicating portfolio. A disadvantage of such a transaction costs function is the fact that it may attain negative values when the amount of transactions exceeds the critical value $\xi = C_0/\kappa$. In the model studied by Amster *et al.* [1] the volatility function has the following form:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 \left(1 - \mathrm{Le}\,\mathrm{sgn}\left(S\partial_S^2 V\right) + \kappa S\partial_S^2 V\right). \tag{7}$$

The nonconstant transaction cost model has been generalized to more realistic transaction cost function by Ševčovič and Žitňanská in the recent paper [25].

In [3] Bakstein and Howison investigated a parametrized model for liquidity effects arising from the asset trading. In their model the volatility function is a quadratic function of the term $S\partial_S^2 V$:

$$\begin{aligned}\sigma(S\partial_S^2 V)^2 = &\sigma_0^2 \Bigg(1 + \gamma^2(1-\alpha)^2 + 2\lambda S\partial_S^2 V + \lambda^2 (1-\alpha)^2 \left(S\partial_S^2 V\right)^2 \\ &+ 2\sqrt{\frac{2}{\pi}}\gamma\,\mathrm{sgn}\left(S\partial_S^2 V\right) + 2\sqrt{\frac{2}{\pi}}\lambda(1-\alpha)^2\gamma\left|S\partial_S^2 V\right|\Bigg).\end{aligned} \tag{8}$$

The parameter λ corresponds to a market depth measure, i.e. it scales the slope of the average transaction price. The parameter γ models the relative bid–ask spreads and it is related to the Leland number through relation $2\gamma\sqrt{2/\pi} = \mathrm{Le}$. Finally, α transforms the average transaction price into the next quoted price, $0 \leq \alpha \leq 1$.

Notice that if additional model parameters (e.g., Le, $\mu, \kappa, \gamma, \lambda$) are vanishing then all the aforementioned nonlinear models are consistent with the original Black–Scholes equation, i.e. $\sigma = \sigma_0$. Furthermore, for call or put options, the function V is convex in the S variable.

The main purpose of this paper is to investigate qualitative and quantitative behavior of a solution to the problem of pricing American style of perpetual put options. We assume the option price is a solution to a stationary generalized Black-Scholes equation with a nonlinear volatility function. We prove existence and uniqueness of a solution to the free boundary problem. We derive a single implicit equation for the free boundary position and the closed form formula for the option price. It is a generalization of the well-known explicit closed form solution derived by Merton for the case of a constant volatility. We also present results of numerical computations of the free boundary position, option price and their dependence on model parameters.

The paper is organized as follows. In the next section we recall mathematical formulation of the perpetual American put option pricing model. We furthermore present the explicit solutions for the case of the constant volatility derived by Merton. In Section 3 we prove the existence and uniqueness of a solution to the free boundary problem. We derive a single implicit equation for the free boundary position ϱ and the closed form formula for the option price. The first order expansion of the free boundary position with respect to the model parameter is also derived. We construct suitable sub– and supper–solutions based on Merton's explicit solutions. In Section 4 we present results of numerical computations of the free boundary position, option price and their dependence on the model parameter.

2 Perpetual American put options

In this section we analyze the problem of pricing the so-called perpetual put options. By definition, perpetual options are options with a very long maturity $T \to \infty$. Notice that both the option price and the early exercise boundary position depend on the remaining time $T - t$ to maturity only. Suppose that there exists a limit of the solution V and early exercise boundary position S_f for the maturity $T \to \infty$. Recently, stationary solutions to generalized Black–Scholes equation have been investigated by Grossinho *et al.* in [12, 13].

For American style put option the limiting price $V = V(S) = \lim_{T-t\to\infty} V(S,t)$ and the limiting early exercise boundary position $\rho = \lim_{T-t\to\infty} S_f(t)$ of the perpetual put option is a solution to the stationary nonlinear Black–Scholes partial differential equation:

$$\frac{1}{2}\sigma(S\partial_S^2 V)^2 S^2 \partial_S^2 V + rS\partial_S V - rV = 0, \qquad S > \varrho, \tag{9}$$

and

$$V(\varrho) = E - \varrho, \quad \partial_S V(\varrho) = -1, \quad V(+\infty) = 0. \tag{10}$$

The purpose of this paper is to analyze the system of equations (9)–(10). In what follows, we will prove the existence and uniqueness of a solution pair $(V(\cdot), \varrho)$ to (9)–(10).

In the rest of the paper, we will assume the function

$$\mathbb{R}_0^+ \ni H \mapsto \sigma(H)^2 H \in \mathbb{R}_0^+ \tag{11}$$

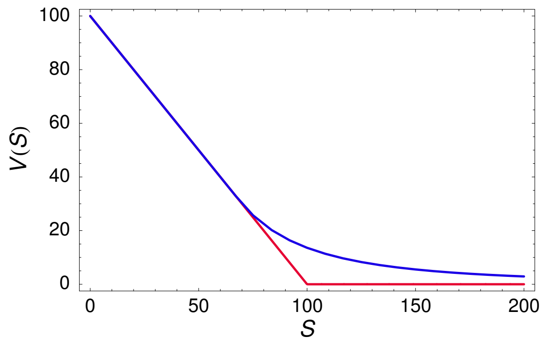

Figure 1: A plot of the price $V(S)$ of a perpetual American put option for the parameters: $E = 100, r = 0.1$ and constant volatility $\sigma_0 = 0.3$ and $\gamma = 2r/\sigma_0^2$.

is nondecreasing and $\sigma(0) > 0$. Then there exists the inverse function $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\frac{1}{2}\sigma(H)^2 H = u \qquad \text{iff} \qquad H = \beta(u). \tag{12}$$

which is an C^1 continuous and nondecreasing function such that $\beta(0) = 0$. Notice that the transformation $H = S\partial_S^2 V$ is a useful tool when analyzing nonlinear generalizations of the Black–Scholes equations. For example, using this transformation the fully nonlinear Black–Scholes equation with a volatility function $\sigma = \sigma(S\partial_S^2 V)$ can be transformed into a quasilinear equation for the new variable H (see [16] and [24] for details).

Remark 1 *Typically, the nonlinear volatility function $\sigma(H)$ is an increasing function satisfying the bounds:*

$$0 < \sigma_0^2 \le \sigma(H)^2 \le \sigma_0^2(1 + \mu H^a)$$

for some constants $\sigma_0 > 0$ and $\mu, a \ge 0$. Then it is easy to verify that, for any $U_0 > 0$, there are constants $M_0, M_1 > 0$ such that

$$M_0 u \le \beta(u) \le M_1 u \quad \textit{for } 0 \le u \le U_0, \quad M_0 u^{\frac{1}{1+a}} \le \beta(u) \le M_1 u \quad \textit{for } u \ge U_0. \tag{13}$$

The estimates (13) imply that the integral

$$\int_{U_0}^{\infty} \frac{\beta(u)}{u} du = +\infty$$

is divergent.

2.1 The Merton explicit solution for the constant volatility case

In the case of a constant volatility $\sigma \equiv \sigma_0$ the free boundary value problem (9)–(10) for the function V and the limiting early exercise boundary position ϱ has a simple explicit solution discovered by Merton (c.f. Kwok [18]). The closed form solution has the following form:

$$V(S) = \begin{cases} \frac{E}{1+\gamma}\left(\frac{S}{\varrho}\right)^{-\gamma}, & S > \varrho, \\ E - S, & 0 < S \le \varrho, \end{cases} \tag{14}$$

where

$$\varrho = E\frac{\gamma}{1+\gamma}, \qquad \gamma = \frac{2r}{\sigma_0^2}. \tag{15}$$

A graph of a perpetual American put option with the constant volatility is shown in Fig. 1.

3 Existence and uniqueness of solutions

In this section we will focus our attention on existence and uniqueness of a solution of the problem (9)–(10).

3.1 Explicit formula for the perpetual American put option price

Since β is the inverse function to $\frac{1}{2}\sigma(H)^2 H$ the pair $(V(\cdot), \varrho)$ is a solution to (9) if and only if

$$S\partial_S^2 V = \beta(rV/S - r\partial_S V).$$

Let us introduce the following transformation of variables:

$$U(x) = r\frac{V(S)}{S} - r\partial_S V(S) = -rS\partial_S\left(\frac{V(S)}{S}\right), \qquad \text{where} \quad x = \ln S. \tag{16}$$

Since

$$\partial_x U(x) = \partial_S\left(rV(S)/S - r\partial_S V\right)\frac{dS}{dx} = -rS\partial_S^2 V(S) + rS\partial_S\left(\frac{V(S)}{S}\right)$$

the function $U(x)$ is a solution to the initial value problem

$$\partial_x U(x) = -U(x) - r\beta(U(x)), \qquad x > x_0 = \ln\varrho, \tag{17}$$

$$U(x_0) = \frac{rE}{\varrho}. \tag{18}$$

The latter initial condition easily follows from the smooth pasting conditions $V(\varrho) = E - \varrho$ and $\partial_S V(\varrho) = -1$. Equation (17) can be easily integrated. We have the following result:

Lemma 1 *The solution $U = U(x)$ to the initial value problem (17)-(18) is uniquely given by*

$$U(x) = G^{-1}(-x + x_0), \quad \textit{for} \;\; x > x_0 = \ln\varrho,$$

where

$$G(U) = \int_{U(x_0)}^{U} \frac{1}{u + r\beta(u)}du. \tag{19}$$

The useful properties of the function G are summarized in the following lemma:

Lemma 2 *The function $G : \mathbb{R}_0^+ \to \mathbb{R}$ is nondecreasing and $G^{-1}(0) = rE/\varrho$. Suppose there exist constants $M_0, M_1, U_0 > 0$ such that*

- *$\beta(u) \le M_1 u$ for all $u \ge U_0$. Then $G(+\infty) = +\infty$;*

- $\beta(u) \geq M_0 u$ *for all* $u \leq U_0$*. Then* $G(0) = -\infty$*,*

where $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is the inverse function to the function* $\mathbb{R}_0^+ \ni H \mapsto \sigma(H)^2 H \in \mathbb{R}_0^+$*.*

Henceforth we will assume that the inverse function β satisfies the growth assumptions from Lemma 2 and so $G(+\infty) = +\infty, G(0) = -\infty$ and, consequently, $G^{-1}(-\infty) = 0$.

Since

$$-rS\partial_S \left(\frac{V(S)}{S} \right) = U(\ln S) = G^{-1}(-\ln S + \ln \varrho)$$

we obtain, by taking into account the boundary condition $V(+\infty) = 0$, that the solution to equation (9) is given by

$$V(S) = \frac{S}{r} \int_S^\infty G^{-1} \left(-\ln \left(\frac{s}{\varrho} \right) \right) \frac{ds}{s}.$$

Using the substitution $u = G^{-1}(-\ln(s/\varrho))$ we have

$$\frac{ds}{s} = -G'(u)du = -\frac{1}{u + r\beta(u)} du.$$

As $G^{-1}(-\infty) = 0$ the expression for $V(S)$ can be simplified as follows:

$$V(S) = \frac{S}{r} \int_0^{G^{-1}(-\ln(S/\varrho))} \frac{u}{u + r\beta(u)} du. \tag{20}$$

3.2 Equation for the free boundary position

Using the expression (20) we can derive a single implicit integral equation for the free boundary position ϱ. Clearly, $V(\varrho) = E - \varrho$ if and only if

$$E - \varrho = \frac{\varrho}{r} \int_0^{G^{-1}(0)} \frac{u}{u + r\beta(u)} du. \tag{21}$$

As $G^{-1}(0) = \frac{rE}{\varrho}$ we obtain

$$\frac{rE}{\varrho} = r + \int_0^{\frac{rE}{\varrho}} \frac{u}{u + r\beta(u)} du = r + \frac{rE}{\varrho} - r \int_0^{\frac{rE}{\varrho}} \frac{\beta(u)}{u + r\beta(u)} du \tag{22}$$

Therefore the free boundary position ϱ is a solution to the following implicit equation:

$$\int_0^{\frac{rE}{\varrho}} \frac{\beta(u)}{u + r\beta(u)} du = 1.$$

3.3 Main result

In this section we summarize the previous results and state the main result on existence and uniqueness of a solution to the perpetual American put option pricing problem (9)–(10).

Theorem 1 *Suppose that the volatility function* $\sigma : \mathbb{R}_0^+ \to \mathbb{R}^+$ *is* C^1 *smooth,* $\sigma(0) > 0$ *and such that* $H \mapsto \frac{1}{2}\sigma(H)^2 H$ *is increasing and it has the inverse function* β*. Suppose that the inverse function* β *satisfies*

$$\beta(u) \le M_1 u \quad \forall u \ge U_0, \quad \beta(u) \ge M_0 u \quad \forall u \in [0, U_0], \quad \text{and} \quad \int_{U_0}^{\infty} \frac{\beta(u)}{u} du = +\infty$$

for some positive constants $M_0, M_1, U_0 > 0$*. Then the perpetual American put option problem (9)–(10) has a unique solution* $(V(\cdot), \varrho)$ *where the free boundary position* ϱ *is a solution to the implicit equation*

$$\int_0^{\frac{rE}{\varrho}} \frac{\beta(u)}{u + r\beta(u)} du = 1 \tag{23}$$

and the option price $V(S)$ *is given by*

$$V(S) = \frac{S}{r} \int_0^{G^{-1}(-\ln(S/\varrho))} \frac{u}{u + r\beta(u)} du. \tag{24}$$

P r o o f. Equation (22) is equivalent to (23). Hence it suffices to prove that (23) has the unique solution ϱ. For the auxiliary function

$$\phi(y) = \int_0^y \frac{\beta(u)}{u + r\beta(u)} du$$

we have $\phi'(y) > 0, \phi(0) = 0$ and

$$\phi(+\infty) = \int_0^\infty \frac{\beta(u)}{u + r\beta(u)} du \ge \int_{U_0}^\infty \frac{\beta(u)}{u + r\beta(u)} du \ge \frac{1}{1 + rM_1} \int_{U_0}^\infty \frac{\beta(u)}{u} du = +\infty.$$

Hence equation (23) has the unique solution $\varrho > 0$. Clearly, $\varrho < E$ because the right hand side of (21) is positive.

Since the ϱ is a solution to (22) we have $V(\varrho) = E - \varrho$. Moreover, as

$$U(x) = r\frac{V(S)}{S} - r\partial_S V(S) \qquad \text{where} \quad x = \ln S.$$

(see (16)) we obtain, for $x_0 = \ln \varrho$,

$$\partial_S V(\varrho) = \frac{V(\varrho)}{\varrho} - \frac{U(x_0)}{r} = \frac{E - \varrho}{\varrho} - \frac{E}{\varrho} = -1.$$

Hence V is a solution to the perpetual American put option pricing problem (9)–(10), as claimed. ◇

Remark 2 *In the case of a constant volatility function* $\sigma(H) \equiv \sigma_0$ *we have* $\beta(u) = \frac{2}{\sigma_0^2} u$*. It follows from equation (23) that*

$$\varrho = E\frac{\gamma}{1+\gamma}, \quad \text{where } \gamma = \frac{2r}{\sigma_0^2}$$

and

$$V(S) = \frac{S}{r} \int_0^{G^{-1}(-\ln(S/\varrho))} \frac{u}{u + r\beta(u)} du = \frac{S}{r}\frac{1}{1+\gamma} G^{-1}(-\ln(S/\varrho)) = \frac{E}{1+\gamma}\left(\frac{S}{\varrho}\right)^{-\gamma}$$

because $G(U) = \frac{1}{1+\gamma}\ln(U/U(x_0)), U(x_0) = rE/\varrho$*, and so* $G^{-1}(f) = \frac{rE}{\varrho} e^{(1+\gamma)f}$*. Hence the solution is identical with Merton's explicit solution.*

3.4 Sensitivity analysis

In this section we will investigate dependence of the free boundary position on model parameters. We consider the volatility function of the form:

$$\frac{1}{2}\sigma(H)^2 H = \frac{\sigma_0^2}{2}\left(1+\mu H^a\right)H + O(\mu^2)$$

as $\mu \to 0$. Here $a \geq 0$ and $\mu \geq 0$ are model parameters. Our goal is to find the first order expansion of the free boundary position ϱ considered as a function of a parameter μ, i.e. $\varrho = \varrho(\mu)$.

First, we derive expression for the derivative $\partial_\mu \beta$ of the inverse function β. For $H = \beta(u;\mu)$ we have $u = \frac{1}{2}\sigma(\beta(u;\mu))^2\beta(u;\mu)$ and so

$$0 = \partial_\mu\left(\frac{\sigma_0^2}{2}\left(1+\mu H^a\right)H\right) = \frac{\sigma_0^2}{2}\left(1+\mu(a+1)\beta^a\right)\partial_\mu\beta + \frac{\sigma_0^2}{2}\beta^{a+1} + O(\mu)$$

For $\mu = 0$ we have $\beta(u;0) = \frac{2}{\sigma_0^2}u$. Therefore

$$\partial_\mu\beta(u;0) = -(\sigma_0^2/2)^{-(a+1)}u^{a+1}.$$

The derivative of the free boundary position $\varrho = \varrho(\mu)$ can be deduced from the implicit equation (23). We have

$$\begin{aligned} 0 &= \frac{d}{d\mu}\int_0^{\frac{rE}{\varrho(\mu)}} \frac{\beta(u;\mu)}{u + r\beta(u;\mu)}du \\ &= \left.\frac{\beta(u;\mu)}{u+r\beta(u;\mu)}\right|_{u=\frac{rE}{\varrho(\mu)}}\left(-\frac{rE}{\varrho(\mu)^2}\partial_\mu\varrho(\mu)\right) + \int_0^{\frac{rE}{\varrho(\mu)}} \frac{u\partial_\mu\beta(u;\mu)}{(u+r\beta(u;\mu))^2}du \end{aligned}$$

Since, for $\mu = 0$ we have $\varrho(0) = E\gamma/(1+\gamma)$ we conclude

$$\partial_\mu\varrho(0) = -\frac{E}{a+1}\gamma(1+\gamma)^{a-2}.$$

In summary we have shown the following result:

Theorem 2 *If the volatility function $\sigma(H)$ has the form $\frac{1}{2}\sigma(H)^2H = \frac{\sigma_0^2}{2}\left(1+\mu H^a\right)H + O(\mu^2)$ as $\mu \to 0$, where $\mu, a \geq 0$, then the free boundary position $\varrho = \varrho(\mu)$ of the perpetual American put option pricing problem has the asymptotic expansion:*

$$\varrho(\mu) = E\frac{\gamma}{1+\gamma} - \mu\frac{E}{a+1}\frac{\gamma}{(1+\gamma)^{2-a}} + O(\mu^2) \quad \text{as } \mu \to 0.$$

Remark 3 *In the case $a = 0$ we have $\sigma(H)^2 = \sigma_0^2(1+\mu)$ it corresponds to the constant volatility model. Thus $\varrho(\mu) = E\frac{\gamma(\mu)}{1+\gamma(\mu)} = E\frac{1}{1+1/\gamma(\mu)}$ where $\gamma(\mu) = 2r/(\sigma_0^2(1+\mu))$. Hence*

$$\varrho(\mu) = E\frac{1}{1+\frac{\sigma_0^2}{2r}(1+\mu)} \quad \text{and, consequently,} \quad \partial_\mu\varrho(0) = -E\frac{\gamma}{(1+\gamma)^2}$$

as claimed by Theorem 2.

3.5 Comparison principle and Merton's sub– and super–solutions

In this section our aim is to derive sub– and super–solutions to the perpetual American put option pricing problem. Throughout this section we will assume a stronger assumption on monotonicity of the volatility function. Namely, we will assume that the function

$$\mathbb{R}_0^+ \ni H \mapsto \sigma(H)^2 \in \mathbb{R}^+$$

is nondecreasing. Clearly, such an assumption implies that $\mathbb{R}_0^+ \ni H \mapsto \frac{1}{2}\sigma(H)^2 H \in \mathbb{R}_0^+$ is an increasing function.

Let $\gamma > 0$ be positive constant. By V_γ we will denote the explicit Merton solution presented in Section 2.1, i.e.

$$V_\gamma(S) = \begin{cases} \frac{E}{1+\gamma}\left(\frac{S}{\varrho_\gamma}\right)^{-\gamma}, & S > \varrho_\gamma, \\ E - S, & 0 < S \le \varrho_\gamma, \end{cases} \tag{25}$$

where

$$\varrho_\gamma = E\frac{\gamma}{1+\gamma}. \tag{26}$$

It means that the pair $(V_\gamma(\cdot), \varrho_\gamma)$ is the explicit Merton solution corresponding to the constant volatility $\sigma_0^2 = 2r/\gamma$ (see (14)).

Then, for the transformed function $U_\gamma(x)$ we have

$$U_\gamma(x) = -rS\partial_S\left(\frac{V_\gamma(S)}{S}\right) = rE\varrho_\gamma^\gamma e^{-(1+\gamma)x}, \quad \text{for } x = \ln S > x_{0\gamma} = \ln \varrho_\gamma.$$

Clearly,

$$\partial_x U_\gamma + U_\gamma + r\beta(U_\gamma) = r\beta(U_\gamma) - \gamma U_\gamma. \tag{27}$$

Next we will construct a super–solution to the solution U of the equation $\partial_x U_\gamma = -U_\gamma - r\beta(U_\gamma)$ by means of the Merton solution U_γ where $\gamma = \gamma^+$ is the unique root of the equation

$$\gamma^+\sigma(1+\gamma^+)^2 = 2r. \tag{28}$$

Since

$$U_{\gamma^+}(x) \le U_{\gamma^+}(x_{0\gamma}) = \frac{rE}{\gamma^+} = r\frac{1+\gamma^+}{\gamma^+}$$

we obtain

$$\frac{1}{2}\sigma((\gamma^+/r)U_{\gamma^+}(x))^2(\gamma^+/r)U_{\gamma^+}(x) \le \frac{1}{2}\sigma(1+\gamma^+)^2(\gamma^+/r)U_{\gamma^+}(x).$$

By taking the inverse function β we finally obtain

$$\frac{\gamma^+}{r}U_{\gamma^+}(x) \le \beta(U_{\gamma^+}(x)).$$

With regard to (27) we conclude that

$$\partial_x U_{\gamma^+}(x) \ge -U_{\gamma^+}(x) - r\beta(U_{\gamma^+}(x)) \quad \text{for } x > x_{0\gamma^+} = \ln \varrho_{\gamma^+} \tag{29}$$

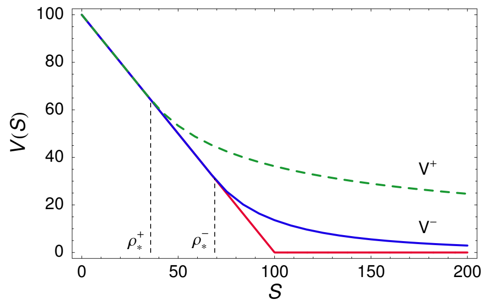

Figure 2: A plot of Merton solutions V^+ and V^- corresponding to constant volatilities $\sigma^+ = 0.6$ and $\sigma^- = 0.3$ for model parameters: $E = 100, r = 0.1$.

Similarly, we will construct a Merton sub–solution U_{γ^-} satisfying the opposite differential inequality. Let γ^- be given by

$$\gamma^- \sigma(0)^2 = 2r, \tag{30}$$

i.e. $\gamma^- = 2r/\sigma(0)^2$. Then

$$U_{\gamma^-} = \frac{1}{2}\sigma(0)^2 \frac{\gamma^-}{r} U_{\gamma^-} \le \frac{1}{2}\sigma((\gamma^-/r)U_{\gamma^-})^2 \frac{\gamma^-}{r} U_{\gamma^-}$$

and so, by taking the inverse function β we obtain $\beta(U_{\gamma^-}) \le \frac{\gamma^-}{r} U_{\gamma^-}$. Then, from (27) we conclude that

$$\partial_x U_{\gamma^-}(x) \le -U_{\gamma^-}(x) - r\beta(U_{\gamma^-}(x)) \quad \text{for } x > x_{0\gamma^-} = \ln \varrho_{\gamma^-}. \tag{31}$$

In Fig. 2 we plot Merton's solutions $V^{\pm}(\cdot)$ corresponding to $\gamma^+ = 0.555$ ($\sigma^+ \doteq 0.6$) and $\gamma^- = 2.222$ ($\sigma^- \doteq 0.3$) where $(\sigma^{\pm})^2 = 2r/\gamma^{\pm}$.

In what follows we will prove the inequalities

$$\varrho_{\gamma^+} \le \varrho \le \varrho_{\gamma^-} \tag{32}$$

where ϱ is the free boundary position for the nonlinear perpetual American put option pricing problem (9)–(10).

Denote

$$\beta^-(u) = \frac{\gamma^-}{r} u$$

the inverse function to the function $H \mapsto \frac{1}{2}\sigma(0)^2 H$. As $\frac{1}{2}\sigma(0)^2 H \le \frac{1}{2}\sigma(H)^2 H$ we have $\beta(u) \le \beta^-(u)$ for any $u \ge 0$. Since

$$\int_0^{\frac{rE}{\varrho}} \frac{\beta(u)}{u + r\beta(u)} du = 1 = \int_0^{\frac{rE}{\varrho_{\gamma^-}}} \frac{\beta^-(u)}{u + r\beta^-(u)} du \ge \int_0^{\frac{rE}{\varrho_{\gamma^-}}} \frac{\beta(u)}{u + r\beta(u)} du$$

we conclude $\varrho \le \varrho_{\gamma^-}$.

On the other hand, let

$$\beta^+(u) = \frac{\gamma^+}{r} u$$

be the inverse function to the function $H \mapsto \frac{1}{2}\sigma(1+\gamma^+)^2 H$. Then, for $u \leq rE/\varrho_{\gamma^+}$ we have

$$H = \beta(u) \leq \beta(rE/\varrho_{\gamma^+}) = \beta(\frac{1}{2}\sigma(1+\gamma^+)^2(1+\gamma^+)) = 1+\gamma^+.$$

Therefore, for $u \leq rE/\varrho_{\gamma^+}$ we have $\beta(u) \geq \beta^+(u)$ and arguing similarly as before we obtain the estimate $\varrho_{\gamma^+} \leq \varrho$ and so the inequalities (32) follows.

For initial conditions we have $U_{\gamma^\pm}(x_{0\gamma^\pm}) = \frac{rE}{\varrho_{\gamma^\pm}}, U(x_0) = \frac{rE}{\varrho}$ and so

$$U_{\gamma^-}(x_{0\gamma^-}) \leq U(x_0) \leq U_{\gamma^+}(x_{0\gamma^+}).$$

Using the comparison principle for solutions of ordinary differential inequalities we have $U_{\gamma^-}(x) \leq U(x) \leq U_{\gamma^+}(x)$. Taking into account the explicit form of the function $V(S)$ from Theorem 1 (see (24)) we conclude the following result:

Theorem 3 *Let $(V(\cdot), \varrho)$ be the solution to the perpetual American pricing problem (9)–(10). Then*

$$V_{\gamma^-}(S) \leq V(S) \leq V_{\gamma^+}(S) \quad \textit{for any} \quad S \geq 0$$

and

$$\varrho_{\gamma^+} \leq \varrho \leq \varrho_{\gamma^-}$$

where $V_{\gamma^\pm}, \varrho_{\gamma^\pm}$ are explicit Merton's solutions corresponding to constant volatilities.

A graphical illustration of the comparison principle is shown in Fig. 6.

4 Numerical approximation scheme and computational results

In this section we propose a simple and efficient numerical scheme for constructing a solution to the perpetual put option problem (9)–(10).

Using transformation $H = \beta(u)$, i.e. $u = \frac{1}{2}\sigma(H)^2 H$ and $du = \frac{1}{2}\partial_H(\sigma(H)^2 H)dH$ we can rewrite the equation for the free boundary position (see (23)) as follows:

$$\int_0^{\beta(rE/\varrho)} \frac{H\frac{1}{2}\partial_H(\sigma(H)^2 H)}{\frac{1}{2}\sigma(H)^2 H + rH} dH = 1. \tag{33}$$

Similarly, the option price (23) can be rewritten in terms of the H variable as follows:

$$V(S) = \frac{S}{r}\int_0^{\beta(G^{-1}(-\ln(S/\varrho)))} \frac{\frac{1}{2}\sigma(H)^2 H \frac{1}{2}\partial_H(\sigma(H)^2 H)}{\frac{1}{2}\sigma(H)^2 H + rH} dH. \tag{34}$$

With this transformation we can reduce computational complexity in the case when the inverse function β is not given by a closed form formula.

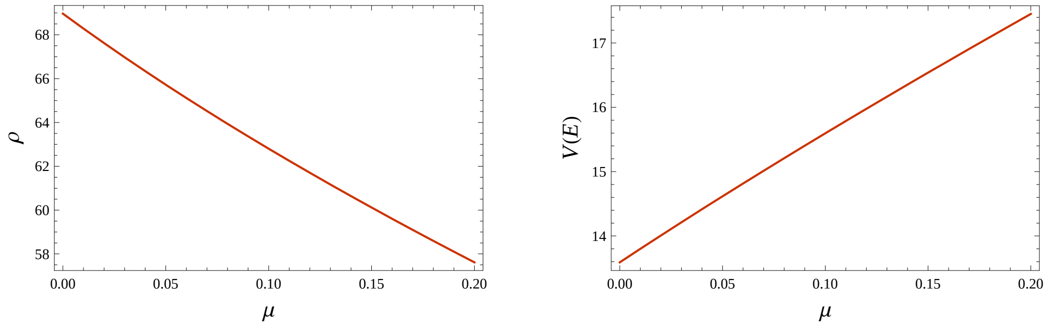

Figure 3: A plot of the dependence of the free boundary position and the perpetual American put option price $V(E)$ for the Frey-Stremme model.

Table 1: The perpetual put option free boundary position $\varrho = \varrho(\mu)$ and the option price $V(S)$ evaluated at $S = E$ for various values of the model parameter $\mu \geq 0$ for the Frey–Stremme model.

μ	0.00	0.01	0.05	0.10	0.15	0.20	0.22
ϱ	68.9655	68.2852	65.7246	62.8036	60.1175	57.6177	56.6627
$V(E)$	13.5909	13.8005	14.6167	15.5961	16.5389	17.4510	17.8083

4.1 Numerical results

Results of numerical calculation for the RAPM model and Frey-Stremme model are summarized in Table 1. We show the position of the free boundary ϱ and the perpetual option value V evaluated at exercise price $S = E$. The results are computed for various values of the model μ for the Frey–Stremme model and the RAPM model. Other model parameter were chosen as: $E = 100, r = 0.1$ and $\sigma_0 = 0.3$.

In the Frey–Stremme model the nonlinear volatility function has the form:

$$\sigma(H)^2 = \sigma_0^2 \left(1 - \mu H\right)^{-2},$$

(see (5)). The range of the parameter μ is therefore limited to satisfy the strict inequality $1 - \mu H = 1 - \mu S \partial_S^2 V(S) > 0$. However, using the identity

$$\frac{1}{1 - \mu H} = 1 + \sum_{n=1}^{\infty} \mu^n H^n.$$

we can approximate the Frey–Stremme volatility function as follows:

$$\sigma(H)^2 = \sigma_0^2 \left(1 + \sum_{n=1}^{N} \mu^n H^n\right)^2, \tag{35}$$

where N is sufficiently large. In computations shown in Fig. 4 and Tab. 2 we present results of the free boundary position and the perpetual American put option price $V(E)$ for $N = 10$ and larger interval of parameter values $\mu \in [0, 8]$. Notice that the results for

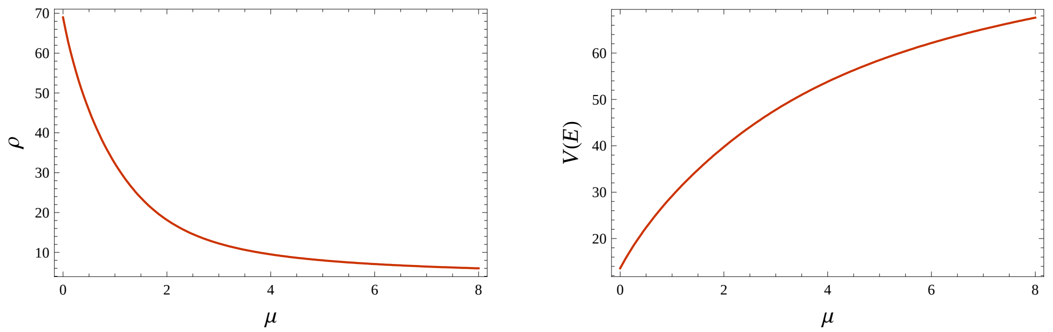

Figure 4: A plot of the dependence of the free boundary position and the perpetual American put option price $V(E)$ for the modified Frey–Stremme model.

Table 2: The perpetual put option free boundary position ϱ and the option price $V(S)$ evaluated at $S = E$ for various values of the model parameter $\mu \geq 0$ of the modified Frey–Stremme model.

μ	0.00	0.10	0.50	1.00	2.00	4.00	8.00
ϱ	68.9655	62.8037	45.3007	31.0862	16.3126	8.3818	5.4556
$V(E)$	13.5909	15.5961	22.4529	29.5719	41.0654	56.1777	70.2259

small values $\mu = 0.1$ computed from the original Frey–Stremme volatility (5) and (35) coincide.

In our next computational example we consider the Risk adjusted pricing methodology model (RAPM). In computations shown in Fig. 5, a) and Tab. 3 we present results of the free boundary position and the perpetual American put option price $V(E)$ for the RAPM model (see Fig. 5, b)). We also show comparison of the free boundary position $\varrho = \varrho(\mu)$ and its linear approximation derived in Theorem 2 (see Fig. 5, c)).

In the last examples shown in Fig. 6 we present comparison of the option price $V(S)$ and the free boundary position ϱ for the Frey–Stremme model (left) and the Risk adjusted pricing methodology model (right) with closed form explicit Merton's solutions corresponding to the constant volatility.

Table 3: The perpetual put option free boundary position ϱ and the option price $V(S)$ evaluated at $S = E$ for various values of the model parameter $\mu \geq 0$ for the RAPM model.

μ	0.00	0.10	0.50	1.00	2.00	4.00	8.00
ϱ	68.9655	66.7331	59.6973	53.3234	44.5408	34.0899	23.6125
$V(E)$	13.5909	14.5761	17.9398	21.3434	26.6857	34.3393	44.1774

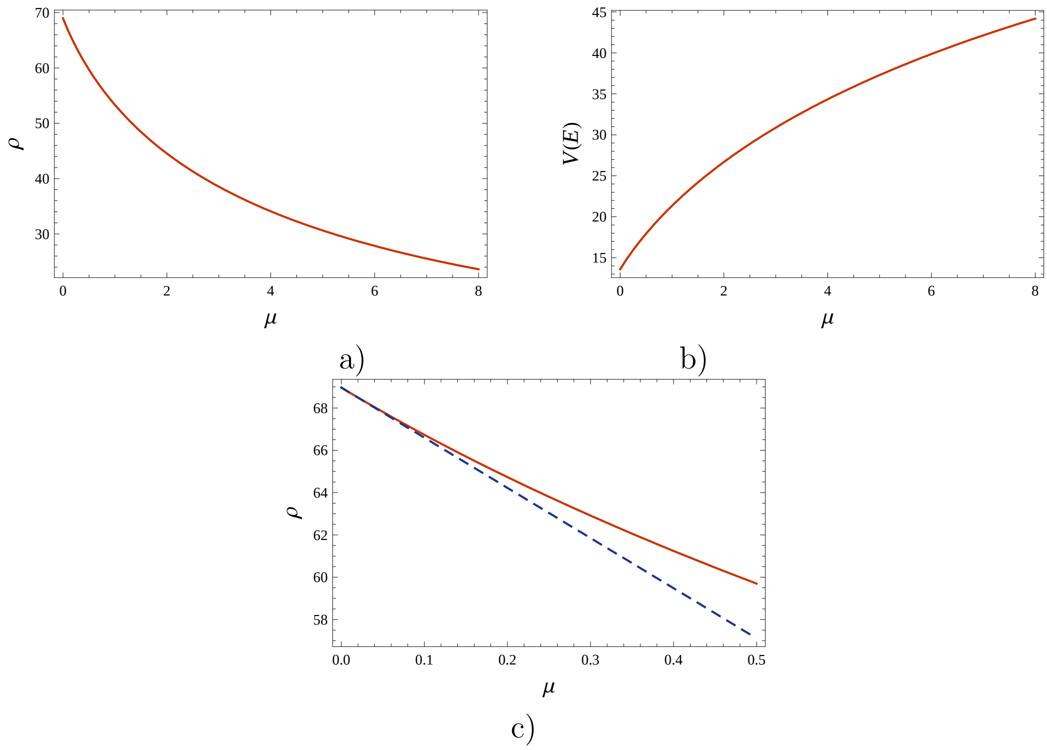

Figure 5: A plot of the dependence of the free boundary position and the perpetual American put option price $V(E)$ for the RAPM model a,b). The comparison of the free boundary position and its linear approximation c).

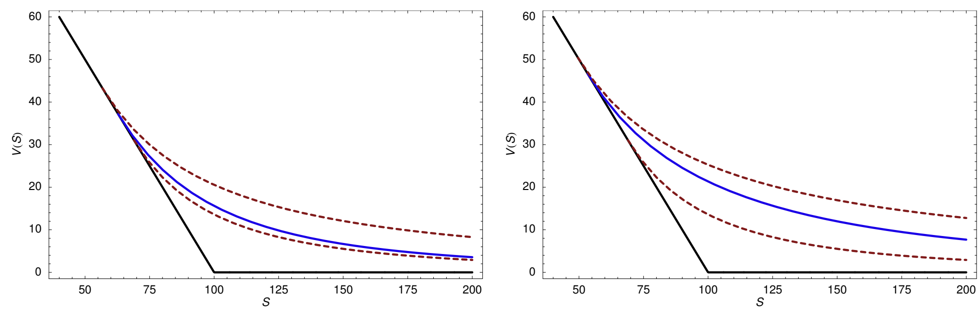

Figure 6: Solid curve represents a graph of a perpetual American put option $S \mapsto V(S)$ for the Frey-Stremme model (left) with $\mu = 0.1$ and the RAPM model (right) with $\mu = 1$. Sub- and super- solutions V_{γ^-} and V_{γ^+} are depicted by dashed curves. The model parameters: $E = 100, r = 0.1$ and $\sigma_0 = 0.3$.

5 Conclusions

In this paper we analyzed the free boundary problem for pricing perpetual American put option when the volatility is a function of the second derivative of the option price. We showed how the problem can be transformed into a single implicit equation for the free boundary position and explicit integral expression for the option price.

Acknowledgements

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