REM WORKING PAPER SERIES

Reputation risk mitigation in investment strategies

Alexandra Moura, Carlos Oliveira

REM Working Paper 0309-2024

February 2024

REM – Research in Economics and Mathematics
Rua Miguel Lúpi 20,
1249-078 Lisboa,
Portugal

ISSN 2184-108X

REM – Research in Economics and Mathematics

Rua Miguel Lupi, 20
1249-078 LISBOA
Portugal

Telephone: +351 - 213 925 912
E-mail: rem@iseg.ulisboa.pt

https://rem.rc.iseg.ulisboa.pt/

https://twitter.com/ResearchRem

https://www.linkedin.com/company/researchrem/

https://www.facebook.com/researchrem/

Reputation risk mitigation in investment strategies

ALEXANDRA MOURA[‡], CARLOS OLIVEIRA[†‡]

amoura@iseg.ulisboa.pt

carlos.m.d.s.oliveira@ntnu.no

[‡]*ISEG - School of Economics and Management, Universidade de Lisboa and REM-Research in Economics and Mathematics, CEMAPRE Rua do Quelhas,6, 1200-781 Lisbon, Portugal*

[†]*Department of Industrial Economics and Technology Management, NTNU, Norway Alfred Getz' vei 3, 7491 Trondheim, Norway*

February 5, 2024

Abstract

We consider an investment model in which a firm decides to invest in the market, taking into account its future revenue and the possible occurrence of adverse events that may impact its reputation. The firm can buy an insurance contract at the investment time to mitigate reputation risk. The firm decides when to enter the market and the insurance strategy that maximizes its value. We consider three types of insurance contracts and different premium principles. We provide analytical conditions for the optimum and study several numerical examples. Results show that the firm's optimal strategy depends on the risk size, the firm's risk aversion, and the insurance premium.

Keywords: Reputaion Risk, Insurance, Risk Mitigation, Investment Strategies, Real Options.

1 Introduction

Insurance contracts are used by private individuals and companies to protect themselves against the occurrence of adverse events with a financial impact. Thus, such contracts are particularly important for firms' risk management. Earlier justifications for the demand for corporate insurance were based on the preferences of investors, who were generally risk averse and promoted the demand for corporate insurance (c.f. [4]). Authors like Main ([24]) and Mayers and Smith ([25]) argue that risk aversion can only partially justify the purchase of corporate insurance by value-maximizing firms, mainly in corporations with diffuse ownership. There is a large body of literature discussing the determinants of corporate insurance demand (see, for instance, [1, 5, 6, 17, 23, 26], and references therein). Some of the main determinants that drive the demand for corporate insurance are (a) the existence of high bankruptcy costs, (b) the existence of disputes between stockholders and bondholders (the underinvestment problem), or (c) even the comparative advantage in having insurance companies managing the corporate risk of firms.

Recently, practitioners and academics have started paying great attention to reputation risk and corporate reputation, although there is no consensus on the definition of reputation risk (cf. [10, 33]). Consequently, reputation risk has a large scope and includes risks such as operational losses or scandals that bring the firm's name into disrepute (see, for instance, [11] and [19]). We follow [12] and [32], defining reputation risk as the risk of financial losses due to the behavior of a firm's stakeholders. Although there is a notorious difficulty in estimating financial losses due to scandals, literature t proposes empirical methods to estimate such costs (cf. [2, 19]). Some authors also argue that the consequences of damages to the firm's reputation can lead to future losses due to an increase in marketing costs or a decrease in revenues ([29]). The importance of this risk in the banking and insurance business is highlighted in the Solvency and Basel regulatory frameworks.

Insurance companies have responded to this reality by introducing insurance policies for corporate reputation risk. Insurers present several policies with different characteristics, as discussed in [13]. These authors also discuss the difficulties in insuring corporate reputation risk, considering challenges such as risk assessment, loss prediction, or demand uncertainty. In this paper, we consider the firm's perspective and analyze the impact of different insurance policies on the firm's investment strategy, assuming that events affecting the firm's reputation may occur. We use the Real Options Approach (ROA) to analyze the problem. ROA is a valuation technique, such as the Net Present Value (NPV), first presented by Myers [27]. It gained academic importance after the publication of the Black and Scholes model ([3])), which provided the theoretical foundations for ROA. ROA is considered more adequate than NPV to evaluate projects for which cash flows are uncertain (see, for instance, [9]).

ROA has been used by academics and practitioners to evaluate different firm's options, such as the abandonment option (see for instance [15, 16]), the investment option ([7, 8]), the option to switch between different production modes ([14, 34], among others). In the context of the present work, it is worth noting that ROA has also been used to define risk management strategies in firms (see, for instance, [21, 22]). In these two papers, the authors study the optimal time when firms should take preventive actions to reduce the number of adverse occurrences that will impact their future cash flows. In addition to the optimal moment, the authors also discuss the investment size in two different contexts. This is especially important if investing in human training and R&D may decrease the frequency or the impact of adverse events in the firm's cash flows. To our knowledge, there are no papers analyzing how insurance may impact a firm's investment decision using the ROA. We find that in most cases, firms benefit from buying insurance, which results in a larger value function and in small investment thresholds. Insurance is less relevant when the reputation risk is smaller or when the firm is less risk-averse.

The paper is organized into five sections and four appendices. In Section 2, we introduce the model, and in Section 3, we present its solution. The insurance optimal strategy is discussed in Section 4, where we provide the optimality conditions and numerical examples. Conclusions are in Section 5. In the appendices, we provide the mathematical proofs, auxiliary results, and additional numerical cases.

2 The investment model

We consider a monopolistic firm producing a single commodity. The firm's revenue at time t is represented by a geometric Brownian motion (gBm) X_t that satisfies the stochastic differential equation (SDE)

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad \text{with} \quad X_0 = x \in (0, +\infty), \tag{2.1}$$

where $\{W_t : t \geq 0\}$ is a Brownian motion.

There are adverse events that can reduce the firm's future revenue. The time of occurrence of such events and their severity in the revenue reduction are random. These negative events are modeled by means of a compound Poisson process. Let N_t represent the number of negative occurrences until time t, which is modeled by a Poisson process with intensity λ and $N_0 = 0$. The processes N_t and W_t are independent and defined in a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, satisfying the usual right continuity and augmentation by $P-$negligible sets.

We assume that every adverse occurrence i will decrease the firm's revenue by a random percentage U_i. The sequence $\{U_i\}_{i\in\mathbb{N}}$ is assumed to be independent and identically distributed to the random variable U, with mean $u \in (0, 1)$. The random variable U is independent of both N_t and W_t. To reduce the effect of each event, the firm has the option of buying an insurance contract, which allows the firm to transfer part of its losses to the insurance company, $h(U) = U - z(U)$, and to retain the remaining part $z(U)$, with $0 \leq z(U) \leq U$. The function $z : [0, 1] \to [0, 1]$ characterizes the insurance contract bought by the firm. The insurance company defines a premium based on the loss of each event, as follows

$$P = E(h(U)) + \alpha M(h(U)), \tag{2.2}$$

where $\alpha M(h(U))$ is the premium loading, which is defined by the insurance company, with $\alpha > 0$ and $M(h(U))$ representing a function of moments of $h(U)$. The premium loading characterizes the premium principle. In this case, we consider premium principles based on moments. Other premium principles, based on other measures, may also be considered (see, for instance, [18]).

A firm producing in the time interval $[t_1, t_2]$ gains an amount given by

$$\int_{t_1}^{t_2} e^{-rt} \left(X_t \left(1 - \sum_{i=0}^{N_{t-\tau}} \Big(z(U_i) + f(z(U_i)) + E(h(U)) + \alpha M(h(U)) \Big) \right) - \theta \right) dt, \tag{2.3}$$

where r is the discount rate, θ is the fixed cost of production, and f is a function that represents how averse the firm is to risk. This function accounts for the cost that the firm may consider to have on top of the expected financial losses. An example of this is the reputation risk, where this function can account for unexpected losses associated with scandals.

If the firm invests at time τ (an $\mathcal{F}_t-$stopping time), its expected value is given by

$$
\begin{aligned}
&J(x,\tau,z) = \\
&= E_x\left[\int_\tau^\infty e^{-rt}\left(X_t\left(1-\sum_{i=0}^{N_{t-\tau}}\Big(z(U_i)+f(z(U_i))+E(h(U))+\alpha M(h(U))\Big)\right)-\theta\right)dt - e^{-r\tau}K\right] \\
&= E_x\left[e^{-r\tau}E_{X_\tau}\left[\int_0^\infty e^{-rt}\left(X_t\left(1-\sum_{i=0}^{N_t}\Big(U_i+f(z(U_i))+\alpha M(h(U_i))\Big)\right)-rK-\theta\right)dt\right]\right], \\
&= E_x\left[e^{-r\tau}\tilde{g}(X_\tau,z)\right], \qquad (2.4)
\end{aligned}
$$

where K is the investment cost and $E_x[\cdot]$ represents $E[\cdot|X_0=x]$. Note that the function $\tilde{g}$ can be written as

$$
\begin{aligned}
\tilde{g}(x,z) =& E_x\left[\int_0^\infty e^{-rt}\left(X_t\left(1-\sum_{i=0}^{N_t}\Big(U_i+f(z(U_i))+\alpha M(h(U_i))\Big)\right)-rK-\theta\right)dt\right] \\
=& \int_0^\infty e^{-rt}\, E_x(X_t)\,\Big(1-\lambda t\big(u+E(f(z(U)))+\alpha M(h(U))\big)\Big)dt - \left(K+\frac{\theta}{r}\right).
\end{aligned}
\qquad (2.5)
$$

We note that the premium paid by the firm to the insurance company also depends on the expected number of negative occurrences λ.

When $f(z)=0$, the firm is risk neutral and assesses its risk only based on the expected value. In this case, it is never optimal to buy insurance since the expected value of the firm will always decrease with insurance. Indeed, insurance has a cost which, in this case, will not, on average, be reflected in a risk reduction to the company. This is due to the fact that the company value is assessed through the expected value, while the insurance premium must be no less than the expected value of the risk, otherwise the insurance company will eventually ruin (c.f. [18]).

Before we state the next assumption, we introduce the set of functions $\mathcal{Z} := \{z : [0,1] \to [0,1] \,|\, z(u) \le u\} \cap \Xi$, where Ξ may represent a particular class of functions parameterized by a set of parameters. Two of the most common classes of contracts are given as follows

$$
z_1(u) = au, \text{ with } a \in [0,1]; \qquad z_2(u) = \begin{cases} u, & u < m \\ m, & m \le u \le l \\ u-l+m, & l < u \end{cases}, \text{ with } 0 \le m < l \le 1. \qquad (2.6)
$$

The parameter a will be referred to as the percentage of risk retained, m as the deductible of the contract, and l as the limit. For the proportional insurance contract, z_1, the insurer is liable for $(1-a)\times 100\%$ of the losses. On the other hand, if the firm buys the insurance contract z_2, the insurer is only liable for losses between $m \times 100\%$ and $l \times 100\%$ of the firm's value. The remaining amount of the losses is covered by the firm. It is worth noting that there are contracts where there is no deductible, $m=0$, or no limit, $l=1$.

Let $\mathcal{T}$ be the set of all $\mathcal{F}_t-$stopping times. Then, our goal is to find both the optimal time, τ^*, to start the project and the optimal insurance contract, z^*, to protect the firm against future occurrences, assuming that its risk preferences are modeled by the function f and that the premium paid to the insurer is characterized by $\alpha M(h(U))$. This is the same as finding the value function V obtained when we solve the optimal stopping problem:

$$V(x) = \sup_{\tau\in\mathcal{T},\, z\in\mathcal{Z}} J(x,\tau,Z) = \sup_{\tau\in\mathcal{T},\, z\in\mathcal{Z}} E[e^{-r\tau}\tilde{g}(X_\tau, z)] = \sup_{\tau\in\mathcal{T}} E[e^{-r\tau} g(X_\tau)], \tag{2.7}$$

where $g(x) = \tilde{g}(x, z^*)$. In order to guarantee that the control problem (2.7) is well-posed, we have to impose additional assumptions on the parameters. The next assumption guarantees that the functional J is well-defined and the optimal contract, z^*, exists.

Assumption 2.1. *The function $f : [0,1] \to \mathbb{R}^+$ is increasing, and both $E(f(U))$ and $M(U)$ are finite. Additionally, U, $M(\cdot)$ and $f(\cdot)$ are such that there is $z^* \in \mathcal{Z}$ verifying*

$$z^* = \arg\min_{z\in\mathcal{Z}} \left(E(f(z(U))) + \alpha M(U - z(U))\right)$$

and

$$E_x\left[\int_0^\infty e^{-rt}\left|X_t\left(1 - \sum_{i=0}^{N_t}\left(U + f(z^*(U_i)) + \alpha M(U_i - z^*(U_i))\right)\right) - rK - \theta\right| dt\right] < \infty,$$

for all $x > 0$.

Given that the expected value of X_t is $e^{\mu t}$, the function g is well-defined for all values of $x > 0$ and z, satisfying Assumption 2.1, when $r > \mu$. After some calculations, the function $\tilde{g}$ can be represented as

$$\tilde{g}(x,z) = \frac{x}{r-\mu}\left(1 - \frac{\lambda}{r-\mu}\left(u + E(f(z(U))) + \alpha M(U - z(U))\right)\right) - \left(K + \frac{\theta}{r}\right). \tag{2.8}$$

Note that the function $g(x) = \tilde{g}(x, z^*) \leq 0$ for all values of $x > 0$ if and only if

$$1 - \frac{\lambda}{r-\mu}\left(u + E(f(z^*(U))) + \alpha M(U - z^*(U))\right) \leq 0,$$

thus, we only consider the set of parameters for which

$$\frac{r-\mu}{\lambda} > \max_{z\in\mathcal{Z}}\left(u + E(f(z(U))) + \alpha M(U - z(U))\right).$$

To conclude this section, we note that from Equation (2.5), the firm considers the expected value of the risk function in the investment decision. This means that if f is a linear function, i.e., $f(u) = \nu u$ for $\nu > 0$, the firm makes the investment decision based on the expected loss. Obviously, if $g(x, z^*) = g(x) \leq 0$, for all $x > 0$, then it is never optimal to invest under this condition. Therefore, when the premium principle is the expected value, buying insurance will only be profitable if the premium is lower than the expected losses measured by the firm. The firm will not transfer the risk when it is cheaper to keep it, i.e., if $\nu < \alpha$. In case $\nu > \alpha$, the reverse situation occurs. The case $\nu = \alpha$ is less interesting because the firm is indifferent to retaining or transferring the risk. This leads us to the next proposition.

Proposition 2.1. *Consider the expected value premium principle and the risk function $f(z) = \nu z$. Then, the optimal contract is such that*

a) if $\nu > \alpha$, then $z^*(u) = 0$ and $h^*(u) = u$;

b) if $\nu < \alpha$, then $z^*(u) = u$ and $h^*(u) = 0$;

We restrict our analysis to the set of parameters for which $g(x) > 0$ for some $x > 0$.

3 The model solution

To find the solution of our optimal stopping problem (see [30]), we will study the correspondent Hamilton-Jacobi-Bellman (HJB, for short) equation, given by

$$\min \{rv(x) - \mathcal{L}v(x), v(x) - g(x)\} = 0, \tag{3.1}$$

where $\mathcal{L}$ represents the infinitesimal generator of the process X_t. For a function $v \in C^2$, with compact support, the infinitesimal generator is defined by (see [28]):

$$\mathcal{L}v(x) \;=\; \lim_{t \searrow 0} \frac{E_x(v(X_t)) - v(x)}{t} \;= \mu x v'(x) + \frac{\sigma^2 x^2}{2} v''(x).$$

Equation (3.1) is a free-boundary problem because one has to solve the ordinary differential equation (ODE)

$$rv(x) - \mathcal{L}v(x) = 0 \tag{3.2}$$

in a domain that is *a priori* unknown. Thus, solving the HJB equation relies on finding two important regions: the waiting region, in which the firm should continue waiting until the moment to start investing, and where Equation (3.2) is active,

$$\mathcal{C} = \{x \in [0, \infty] \, : v(x) > g(x)\},$$

and the investment region, in which the firm should make the decision of investment,

$$\mathcal{S} = \{x \in [0, \infty] \, : v(x) = g(x)\}.$$

A verification result providing sufficient conditions for a solution of the HJB equation to be the value function can be found in the appendix. We note that the function g is increasing, allowing us to guess that

$$\mathcal{C} = (0, x^*) \quad \text{and} \quad \mathcal{S} = [x^*, \infty),$$

where x^* is the investment threshold, which will be defined in the proposition below. This means that while the process X is in the continuation region, i.e. $0 < X < x^*$, the firm should wait to invest, and as soon as the process reaches the level x^*, the firm invests immediately.

The value function in the waiting region is the solution of the ODE (3.2), which has the following general form

$$v(x) = Ax^{\beta_1} + Bx^{\beta_2}, \quad x > 0,$$

where β_1 and β_2 are the roots of the characteristic polynomial $r - \mu\beta - \frac{\sigma^2}{2}\beta(\beta-1) = 0$. It is a matter of trivial calculations to get the expressions:

$$\beta_1 = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} > 1,$$
$$\beta_2 = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} < 0.$$

Proposition 3.1. *Let V be the value function defined as in (2.7). Then V is given by:*

$$V(x) = \begin{cases} Ax^{\beta_1}, & 0 < x < x^* \\ g(x), & x \geq x^* \end{cases}. \tag{3.3}$$

The parameter A and the threshold x^* are given by:

$$A = \frac{(x^*)^{1-\beta_1}}{r-\mu}\left(1 - \frac{\lambda}{r-\mu}(u + E(Z(f(U))) + \alpha M(h^*(U)))\right) - \left(K + \frac{\theta}{r}\right)(x^*)^{-\beta_1}, \tag{3.4}$$
$$x^* = \frac{\left(K + \frac{\theta}{r}\right)\beta_1(r-\mu)^2}{(\beta_1 - 1)\left((r-\mu) - \lambda(u + E(f(z^*(U))) + \alpha M(h^*(U)))\right)}. \tag{3.5}$$

Equation (3.3) shows the optimal expected value of the firm. The option value to wait until the decision to invest in the market is here given by the term Ax^{β_1}. When the firm invests, its value is given by $g(x)$. A firm that is waiting to invest will buy the optimal insurance contract at the investment moment. The total value paid by this firm to the insurance company is then given by

$$E_{x^*}\left[\int_0^\infty e^{-rt} X_t \sum_{i=0}^{N_t}\Big(E(h(U)) + \alpha M(h(U))\Big)dt\right] = \frac{\lambda x^*}{r-\mu}\Big(E(h(U)) + \alpha M(h(U))\Big). \tag{3.6}$$

Next, we present a comparative static regarding the uncertainty parameters μ, σ and λ. With such analysis, we may understand how the firm should adapt its investment strategy when there is a sudden change in both the market behavior or the intensity of the adverse occurrences. The following proposition shows the impact of each parameter in the investment decision.

Proposition 3.2. *Let x^* be the threshold defined in Proposition 3.1 by Equation 3.5. Then x^* is monotonically increasing with σ and λ, but it is not monotonic with μ.*

A usual result in real options is that the investment decision is postponed whenever the uncertainty increases. In this case, the investment decision is postponed when σ increases, which represents an increase of the market uncertainty, or when λ increases, which leads to an increase in the expected number of negative occurrences.

The threshold x^* shows an unusual behavior with the drift μ, since it first decreases and then increases. This behavior can be explained by the fact that each negative occurrence reduces the revenue of the firm by a certain percentage, i.e., there is a multiplicative interaction between the compound Poisson process and the gBm. When μ increases, one expects an increase in the revenue, which leads to a decrease in the investment threshold because the firm wants to invest sooner. However, if the current market state provides the firm with a large revenue, any increase in μ may indirectly result in the growth of the absolute loss, which will occur if an adverse event occurs. This loss is not overcompensated by the increase in the revenue. In Figure 1 we display an example where x^* is not a monotonic function of μ.

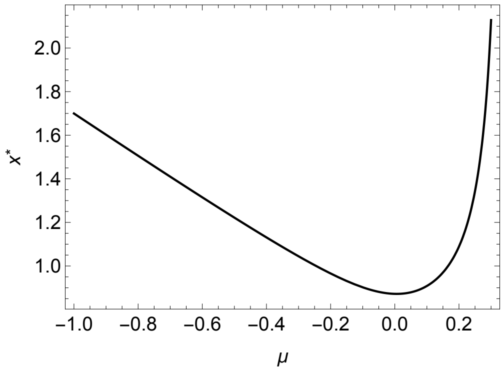


Figure 1: Monotony of x^* as a function of μ, with parameters $\lambda = 0.5$, $r = 0.5$, $\sigma = 0.2$, $u = 0.1$, $E(f(U)) = 0.2$, $\frac{\theta}{r} + K = 1$, without insurance.

The comparative statics is not changed if another insurance contract is considered instead of the optimal insurance treaty. In particular, it is not changed if no insurance is considered. Indeed, the insurance contract is independent of the market conditions since we are considering that X_t is independent of U_i and N_t.

4 The optimal insurance contract

In this section, we analyze the optimality of the insurance contracts and their impact on the investment strategy. To find the optimal insurance policy for the firm, we have to solve the following minimization problem

$$\min_{z \in \mathcal{Z}} \left(E(f(z(U))) + \alpha M(h(U)) \right). \tag{4.1}$$

We present the optimality conditions under which the contracts z_1 and z_2, defined in (2.6), are optimal, considering the cases $m = 0$ or $l = 1$. We write $z_2(u, m)$ when $l = 1$ and $z_2(u, l)$ when $m = 0$. The analysis is performed considering four different risk functions and three premium principles. The risk functions considered in this paper are the following:

$$f_1(z) = \nu z, \quad f_2(z) = \nu z^2, \quad f_3(z) = \nu z 1_{\{z>q\}}, \quad \text{and} \quad f_4(z) = \nu\sqrt{z}, \tag{4.2}$$

where $\nu > 0$. These functions are possible representations of the firm's risk assessment. While f_1 assesses risk in a linear way, f_2, f_3 and f_4 do it in a different manner, weighting more or less the underlying risk. As mentioned in Section 2, we restrict our analysis to premium principles based on moments, namely the expected value, the variance, and the standard deviation premium principles. These premium principles are defined as in Equation (2.2) considering M as the expected value, the variance, and the standard deviation, respectively. In what follows, we will use the notation $u_2 = E(U^2)$, $u_{1/2} = E(U^{1/2})$, $Var(U) = \sigma^2(U)$, and $\sqrt{Var(U)} = \sigma(U)$.

The results regarding the optimal insurance treaty depend on the underlying risk distribution for the proportion of the losses, U, the frequency of the adverse events, λ, the firm risk preference function, f, and the insurance premium, M. Therefore, our analysis is organized into two parts. In Section 4.1, we assume that the insurer always opts for the expected value premium principle, and, therefore, the optimal treaty for the firm depends only on its risk preferences and the type of insurance chosen, given an underlying risk. In Section 4.2, the firm opts to transfer a percentage of its risk, which means that variations on the optimal quantity of risk transferred depend only on the insurance premium and the firm's risk preferences, given an underlying risk.

We illustrate the results below with numerical examples. The optimal insurance treaty does not depend on the market conditions, namely on the market parameters. Nevertheless, it is worth mentioning that the firm's investment decision is impacted by both the market conditions and the optimal insurance treaty. We consider the following market parameters for the numerical results: $\sigma = 0.2$, $\mu = 0.005$, $r = 0.05$ and $I = 1$. For the sake of simplicity, we always consider $\nu = 0.5$ and $\alpha = 0.3$. We consider two different scenarios for the underlying risk that lead to significantly different optimal solutions.

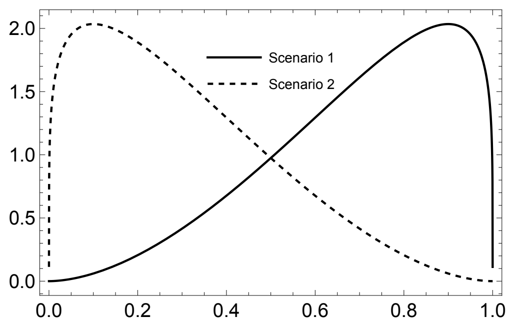
Scenario 1: In the first scenario, the frequency of the adverse events is lower, $\lambda = 0.04$, and the impact of each event is larger, modeled through a Beta distribution with parameters $a = 2.8$ and $b = 1.2$.

Scenario 2: In the second scenario, the frequency of the adverse events is higher, $\lambda = 0.08$, and the size impact of each event is smaller on average, being modeled by means of a Beta distribution with parameters $a = 1.2$ and $b = 2.8$.

The Beta density is parameterized as follows

$$p_U(u) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{1}{u} u^a (1-u)^{b-1}, \quad 0 < u < 1,$$

and Figure 2 depicts the two considered Beta densities for the impact of each event, U, in the two scenarios.



	Scenario 1	Scenario 2
$E[U] = u$	0.7	0.3
$E[U^2] = u_2$	0.532	0.132
$E[\sqrt{U}] = u_{1/2}$	0.825652	0.510476

Figure 2: Right: Density functions of the Beta distributions for each scenario considered. Left: Moments of U used in the different insurance premia and in the optimal solutions.

4.1 Expected value premium principle

In this subsection, we assume that the premium principle is calculated according to the expected value principle, i.e., $P = E(h(U)) + \alpha M(h(U))$, with $M(h(U)) = E(h(U))$. In the next propositions, we find the optimal contracts for each risk function. The case of the linear function $f_1(z) = \nu z$ is in Proposition 2.1.

Proposition 4.1. *Consider the expected value premium principle and the risk function $f_2(z) = \nu z^2$. Then, the optimal contracts in the classes of (i) proportional contracts, (ii) contracts with a deductible, and (iii) contracts with a limit are characterized respectively by*

$$(i) \quad \hat{a} = \begin{cases} \frac{\alpha u}{2\nu u_2}, & \nu > \frac{\alpha u}{2u_2} \\ 1, & \nu \leq \frac{\alpha u}{2u_2} \end{cases}, \qquad (ii) \quad \hat{m} = \begin{cases} \frac{\alpha}{2\nu}, & \nu > \frac{\alpha}{2} \\ 1, & \nu \leq \frac{\alpha}{2} \end{cases},$$

and

$$(iii) \quad \hat{l} = \min_{0 \leq l \leq 1} \left\{ 2\nu \left(\int_l^1 (y-l) S_U(y) dy \right) + \alpha \int_0^l S_U(y) dy \right\}.$$

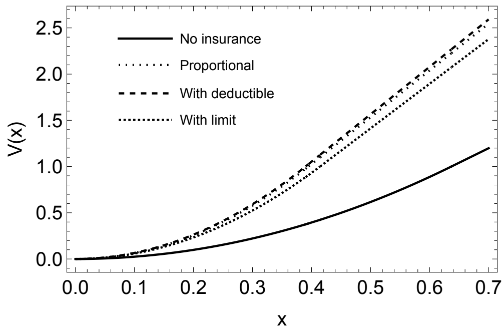
Figure 3 shows the value function, the value of the investment thresholds x^*, the optimal level of insurance to buy, and the total premium the firm pays to the insurance company. In this case, we can observe that the firm benefits from buying the optimal quantity of insurance because its expected value increases even though it pays an insurance premium. This can be observed through the value functions, which are higher for each x when the firm buys the optimal quantity of any of the insurance contracts. Looking at the optimal investment thresholds, we can also conclude that the firm invests earlier in the market if it starts buying these quantities of insurance. The contract with a deductible is the one that increases the most the firm value, although the proportional insurance contract also leads to a very similar firm's value. For the insurance contract with a deductible, the firm accommodates losses up to 30% of the firm's value. For losses greater than this value, the insurer covers the difference. For the proportional contract, the insurer covers 40% of the losses. Finally, for the contract with a limit, the insurer is liable for losses up to 46.5% of the firm's value. The firm's value in the second scenario is shown in Figure 10, in the Appendix A. In that case, buying insurance does not improve the firm's value much. This is explained by the fact that the adverse events have a small impact on the value of the firm, on average.

Proposition 4.2. *Consider the expected value premium principle and the risk function $f_3(z) = \nu z 1_{\{z>q\}}$. Then, the optimal contracts in the classes of (i) proportional contracts, (ii) contracts with a deductible, and (iii) contracts with a limit are characterized respectively by*

$$(i) \quad \hat{a} = \arg\min_{a \in [q,1]} \left(\nu a \int_{\frac{q}{a}}^1 y f_U(y) dy + \alpha(1-a)u \right), \quad (ii) \quad \hat{m} = \begin{cases} 1, & \nu < \alpha \\ q, & \nu \geq \alpha \end{cases}, \quad \textit{and}$$

$$(iii) \quad \hat{l} = \arg\min_{0 \leq l \leq 1-q} \left(\alpha \int_0^l S_U(y) dy + \nu q S_U(l+q) + \nu \int_{l+q}^1 S_U(y) dy \right).$$

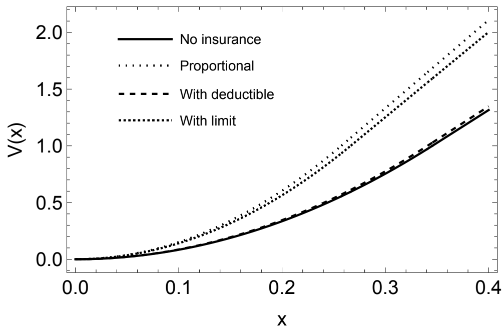
Proposition 4.2 provides expressions to find the optimal quantity of insurance the firm buys to maximize its value when it is mainly concerned with risks that may lead to large unexpected



Contract	x^*	optimal value	premium
No insurance	0.636792	—	—
Proportional	0.394818	$\hat{a}$ =0.394737	0.193299
With deductible	0.389766	$\hat{m}$ =0.3	0.181794
With limit	0.414386	$\hat{l}$ =0.465	0.213703

Figure 3: Illustration of the results from Proposition 4.1 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

losses. The solution of this case is highly dependent on the size of the insurance claim, which is represented here by the U density and survival functions. Figure 4 depicts the firm's optimal value for Scenario 1 and presents the optimal quantities associated with the solution. We can also observe the optimal investment thresholds and insurance contracts. In this case, the proportional contract is the best option for the firm, while the contract with a deductible provides a solution very similar to the case with no insurance. We note that the cost of buying an insurance contract with a deductible is 0.0081, while the cost of a proportional contract is 0.031. Nevertheless, it is optimal to buy proportional insurance. As in the previous case, buying insurance increases the value of the firm on average. Looking at Figure 11, in Appendix A, one can conclude that the firm has no incentive to buy insurance when the loss size is small.



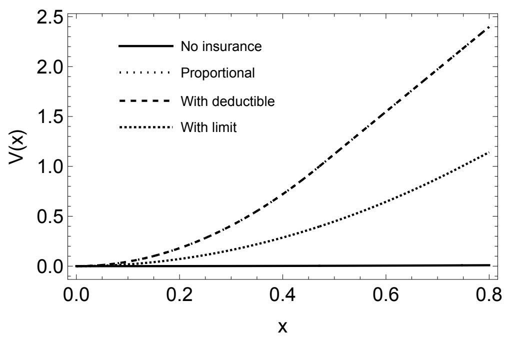
Contract	x^*	optimal value	premium
No insurance	0.345496	—	—
Proportional	0.257306	$\hat{a}$ =0.85	0.0312198
With deductible	0.340729	$\hat{m}$ =0.85	0.00807249
With limit	0.266391	$\hat{l}$ =0.15	0.0460926

Figure 4: Illustration of the results from Proposition 4.2 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

Proposition 4.3. *Consider the expected value premium principle and the risk function $f_4(z) = \nu\sqrt{z}$. Then, the optimal contracts in the classes of (i) proportional contracts, (ii) contracts with a deductible, and (iii) contracts with a limit are characterized respectively by*

$$(i) \quad \hat{a} = \begin{cases} 0, & \nu \geq \frac{\alpha u}{u_{1/2}} \\ 1, & \nu < \frac{\alpha u}{u_{1/2}} \end{cases}, \quad (ii) \quad \hat{m} = \begin{cases} 0, & \nu \geq \frac{\alpha u}{u_{1/2}} \\ 1, & \nu < \frac{\alpha u}{u_{1/2}} \end{cases}, \quad (iii) \quad \hat{l} = \begin{cases} \frac{\nu^2}{4\alpha^2}, & \nu < 2\alpha \\ 1, & \nu \geq 2\alpha \end{cases}.$$

In Proposition 4.3, the firm assesses its risk using a square root function. Since the losses are measured as a percentage of the firm's value, this risk function amplifies the impact of their unexpected losses. Consequently, the firm's value with no insurance is much smaller than the firm's value when the optimal quantity of insurance for any contract is considered, as seen in Figure 5. Losses are so relevant that the firm is willing to pass all the risk to the insurer in the proportional contract and in the contract with a deductible. Therefore, the optimal threshold of investment and the total premium paid by the firm are equal in these two contracts. The results for the second scenario are similar, although the original losses are smaller than in the first scenario, as illustrated in Figure 12, in Appendix A.



Contract	x^*	optimal value	premium
No insurance	8.31707	—	—
Proportional	0.47093	$\hat{a}$ =0	0.38093
With deductible	0.47093	$\hat{m}$ =0	0.38093
With limit	0.74719	$\hat{l}$ =0.694444	0.528608

Figure 5: Illustration of the results from Proposition 4.3 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

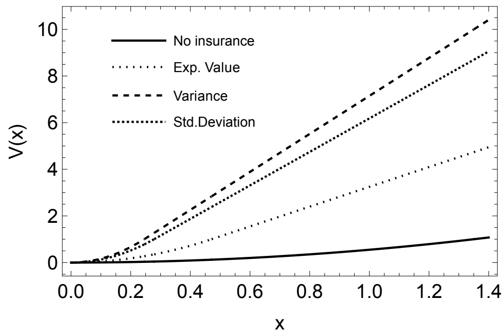
4.2 The proportional contract

In this subsection, we assume that the firm intends to buy a proportional contract to protect itself against unpredictable risk, i.e., $z_1(U) = aU$, with $0 < a < 1$. In the next propositions, we will present the optimal contract for each risk function and premium principle.

Proposition 4.4. *Consider the quota-share contract and the risk function $f_1(z) = \nu z$. The optimal level of risk that should be transferred for the insurer, $\hat{a}$, when the premium principle is based on* *the (i) expected value, (ii) variance, and (iii) standard deviation is given by*

$$(i)\;\; \hat{a} = \begin{cases} 1, & \nu < \alpha \\ 0, & \nu > \alpha \end{cases}, \quad (ii)\;\; \hat{a} = \begin{cases} 1 - \frac{\nu u}{2\alpha\sigma^2(U)}, & \nu < \frac{2\alpha\sigma^2(U)}{u} \\ 0, & \nu > \frac{2\alpha\sigma^2(U)}{u} \end{cases}, \quad (iii)\;\; \hat{a} = \begin{cases} 1, & \nu < \frac{\alpha\sigma(U)}{u} \\ 0, & \nu > \frac{\alpha\sigma(U)}{u} \end{cases}.$$

Figure 6 illustrates the results of Proposition 4.4 for Scenario 1. In this case it is always optimal to transfer all the risk to the insurance company. This is related to the fact that ν is significantly larger than α. However, the total price paid by the firm is different because the insurer uses different premium principles in each case. Thus, the optimal insurance contract is the cheapest one, i.e., the proportional contract with the variance premium principle. Indeed, since losses are measured as a percentage of the firm's value, i.e., with values between zero and one, then the variance is smaller than the expected value of the losses. The results for Scenario 2 are similar and are depicted in Figure 13.



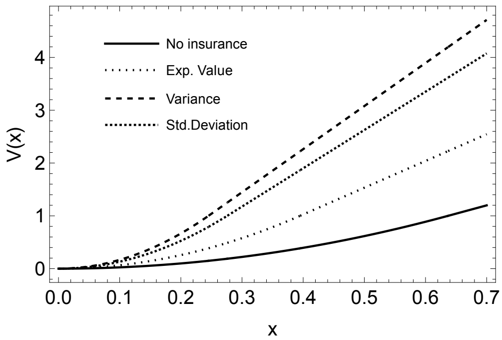
Premium	x^*	$\hat{a}$	premium
No insurance	1.35	—	—
Exp. Value	0.47093	0	0.38093
Variance	0.245514	0	0.155514
Std. Deviation	0.278528	0	0.188528

Figure 6: Illustration of the results from Proposition 4.4 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

Proposition 4.5. *Consider the quota-share contract and the risk function $f_2(z) = \nu z^2$. The optimal level of risk that should be transferred for the insurer, $\hat{a}$, when the premium principle is based on the (i) expected value, (ii) variance, and (iii) standard deviation is given by*

$$(i)\;\; \hat{a} = \begin{cases} \frac{\alpha u}{2\nu u_2}, & \nu > \frac{\alpha u}{2u_2} \\ 1, & \nu < \frac{\alpha u}{2u_2} \end{cases}, \quad (ii)\;\; \hat{a} = \frac{\alpha\, \sigma^2(U)}{\nu u_2 + \alpha\sigma^2(U)}, \quad (iii)\;\; \hat{a} = \begin{cases} \hat{a} = \frac{\alpha\sigma(U)}{2\nu u_2}, & \nu > \frac{\alpha\sigma(U)}{2u_2} \\ 1, & \nu < \frac{\alpha\sigma(U)}{2u_2} \end{cases}.$$

Proposition 4.5 presents the optimal percentage of the losses that the firm retains when the firm's risk function is quadratic. In Figure 7, it can be seen that buying the optimal levels of insurance increases the firm's value. On the other hand, the premium calculated based on the variance premium principle benefits the firm. It is optimal to buy insurance because the size of the losses is relatively high. In Scenario 2, where losses are smaller, the impact of buying insurance is less relevant, as seen in Figure 14.



Premium	x^*	$\hat{a}$	premium
No insurance	0.636792	—	—
Exp. Value	0.394818	0.394737	0.193299
Variance	0.245175	0.0452261	0.148157
Std. Deviation	0.275832	0.115567	0.165127

Figure 7: Illustration of the results from Proposition 4.5 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

Proposition 4.6. *Consider the quota-share contract and the risk function $f_3(z) = \nu z 1_{\{z>q\}}$. The optimal level of risk that should be transferred for the insurer, $\hat{a}$, when the premium principle is based on the (i) expected value, (ii) variance, and (iii) standard deviation is given by*

$$\hat{a} = \arg\min_{a\in[q,1]} \left(\nu a \int_{\frac{q}{a}}^{1} y f_U(y) dy + \alpha M((1-a)U) \right)$$

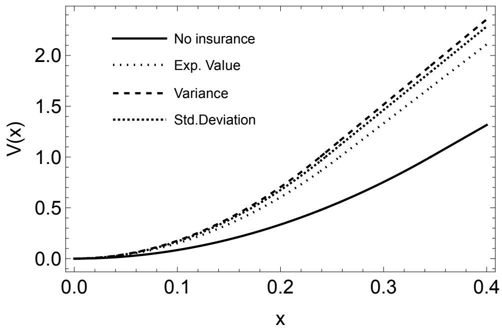
where $M((1-a)U)$ equals (i) $(1-a)u$, (ii) $(1-a)^2\sigma^2(U)$, and (iii) $(1-a)\sigma(U)$.

In Proposition 4.6, we present the equation to calculate the optimal percentage of the loss retained by the firm when it considers function f_3 to assess risk. This risk function considers only large losses. Figure 8 shows that, regardless of the premium principle, it is always optimal to retain 85% of the losses and to transfer the remaining 15%. This mainly happens because we are fixing q at the level 0.85. Although the optimal retaining level is always the same, the optimal time to invest and the amount paid to the insurer is different when the insurer uses different premium principles. In Figure 15, the optimal solution for Scenario 2 is depicted. Since the losses are expected to be smaller in this scenario, the firm has almost no incentive to buy insurance, and consequently, optimal levels $\hat{a}$ are always larger.

Proposition 4.7. *Consider the quota-share contract and the risk function $f_4(z) = \nu\sqrt{z}$. The optimal level of risk that should be transferred for the insurer, $\hat{a}$, when the premium principle is based on the (i) expected value, (ii) variance, and (iii) standard deviation is given by*

$$(i) \quad \hat{a} = \begin{cases} 0, & \nu \geq \frac{\alpha u}{u_{1/2}} \\ 1, & \nu < \frac{\alpha u}{u_{1/2}} \end{cases}, \quad (ii) \quad \hat{a} = \arg\min_{a\in[0,1]} \left\{ \sqrt{a}\nu u_{1/2} + \alpha(1-a)^2\sigma^2(U) \right\},$$

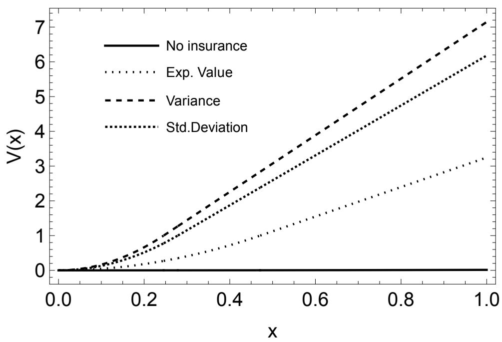
$$(iii) \quad \hat{a} = \begin{cases} 0, & \nu \geq \frac{\alpha\sigma(U)}{u_{1/2}} \\ 1, & \nu < \frac{\alpha\sigma(U)}{u_{1/2}} \end{cases}.$$



Premium	x^*	$\hat{a}$	premium
No insurance	0.345496	—	—
Exp. Value	0.257306	0.85	0.0312198
Variance	0.238394	0.85	0.0223102
Std. Deviation	0.24352	0.85	0.0247248

Figure 8: Illustration of the results from Proposition 4.6 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

In Proposition 4.7, the risk function f_4 is considered and consequently the firm is strongly concerned with the impact of having unexpected losses. Thus, the numerical results in Figure 9 show that the firm decides to pass all its losses to the insurance company, meaning that the price requested by the insurer compensates for its risk. Once more, pricing the insurance contract with the variance premium principle leads to the cheapest option. A similar behaviour can be found in Figure 16, when we consider losses with smaller size.



Premium	x^*	$\hat{a}$	premium
No insurance	8.31707	—	—
Exp. Value	0.47093	0	0.38093
Variance	0.245514	0	0.155514
Std. Deviation	0.278528	0	0.188528

Figure 9: Illustration of the results from Proposition 4.7 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

5 Conclusions

The occurrence of adverse events can have an enormous impact on the profitability of firms. To protect against the impact of such events, firms can buy insurance, which permits them to transfer part of their risk to insurance companies. The insurance contracts may have different characteristics, and their optimality depends on the type of adverse event the firm wants to protect against and how averse the firm is to risk. In this paper, we consider a firm that takes reputation risk into account when evaluating the possibility of investing in the market.

From our analysis, we can conclude that the optimal type of insurance is heavily dependent on the size of the loss as well as on the firm's risk aversion. When buying insurance is optimal, we can notice that the firm is willing to invest sooner than without insurance. To model risk aversion, we include different risk functions in the model. From the numerical examples, we can observe that, in general, the firm benefits from buying proportional insurance when the premium principle used by the insurance company is based on the expected value.

When the size of each occurrence is small, even if they are more frequent, buying insurance has little impact on the value of the firm, especially when the firm is less risk averse (see Figures 10 and 11). However, if the firm is more risk averse, then it can be optimal to buy insurance anyway 12. It is worth mentioning that, when comparing the optimal insurance contracts in the classes of proportional contracts, contracts with a deductible and contracts with a limit, the cheapest option is not always the contract that increases the firm's value most. Finally, we note that the volatility of the revenue postpones the investment decision, while the frequency of adverse events, λ, has a non linear impact on the investment threshold, because it is represented as a percentage of the firm's value. This is in contrast with results in classical real option models, where uncertainty postpones the investment decision.

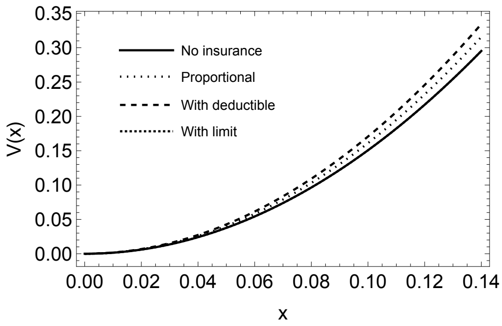
Given that, to the best of our knowledge, this is the first work highlighting the importance of buying insurance in a firm's investment strategy using a real options approach, we keep the model simple to facilitate the analysis. It would be interesting to continue with this research by considering combinations of the insurance treaties analysed in this paper and possible dependencies between the risk and profitability processes. Furthermore, it would be more realistic to consider absolute losses in addition to percentage losses, which would result in a more complex model, both in terms of recovering the value function as well as the optimal insurance strategy.

Acknowledgments

The authors were partially funded by FCT – Fundação para a Ciência e Tecnologia (Portugal), with national funding through research grants CEMAPRE/REM UIDB/05069/2020 and EXPL/EGE-ECO/0886/2021.

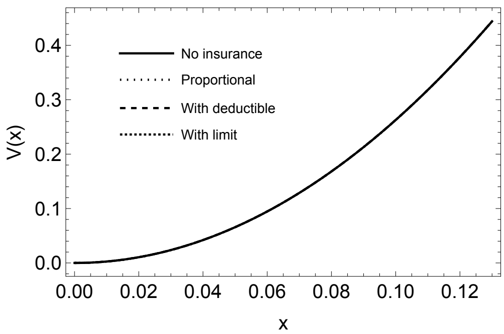
A Appendix: additional figures

Here, we present additional figures depicting the numerical results of Propositions 4.1 to 4.7 regarding Scenario 2.



Contract	x^*	optimal value	premium
No insurance	0.257634	—	—
Proportional	0.249161	$\hat{a}$ =0.681818	0.0549664
Deductible	0.242061	$\hat{m}$ =0.3	0.0473515
With limit	0.257634	$\hat{l}$ =0	0

Figure 10: Illustration of the results from Proposition 4.1 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

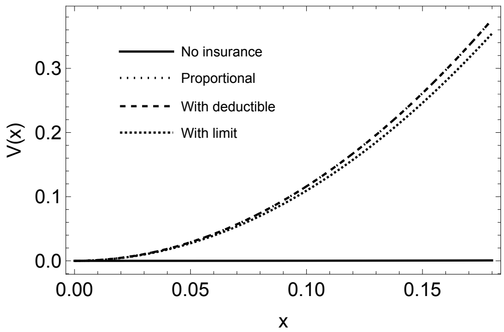


Contract	x^*	optimal value	premium
No insurance	0.195074	—	—
Proportional	0.195074	$\hat{a}$ =1	0
Deductible	0.195034	$\hat{m}$ =0.85	0.000119942
With limit	0.195074	$\hat{l}$ =0	0

Figure 11: Illustration of the results from Proposition 4.2 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

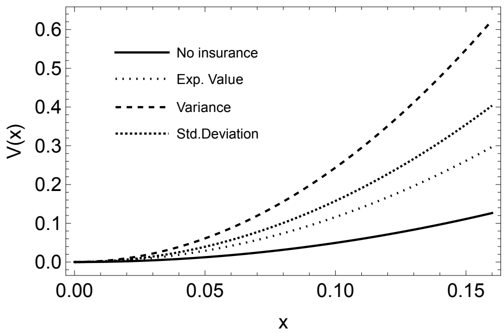
B Appendix: auxiliary results

In this section, we introduce some results and the necessary notation to proceed with the analysis of the insurance contract for the firm. For each type of contract, premium principle, and risk function,



Contract	x^*	optimal value	premium
No insurance	6.97145	—	—
Proportional	0.293478	$\hat{a}$ =0	0.203478
Deductible	0.293478	$\hat{m}$ =0	0.203478
With limit	0.302388	$\hat{l}$ =0.694444	0.206937

Figure 12: Illustration of the results from Proposition 4.3 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

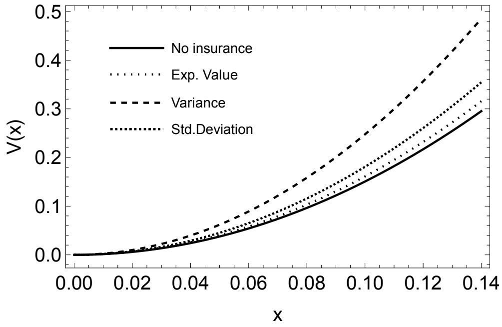


Premium	x^*	$\hat{a}$	premium
No insurance	0.45	—	—
Exp. Value	0.293478	0	0.203478
Variance	0.202581	0	0.112581
Std. Deviation	0.251843	0	0.161843

Figure 13: Illustration of the results from Proposition 4.4 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

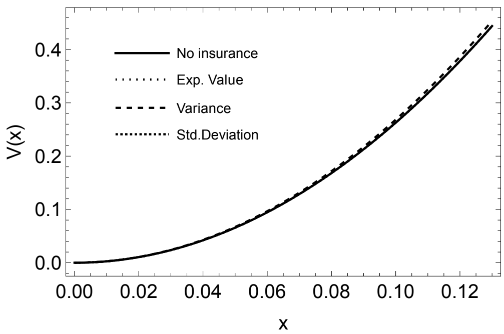
we will present the relevant moments. Most of these results can be found in reference books in the field of risk and insurance, such as [18].

Proposition B.1. *Let U be a random variable with support in the interval $(0, 1)$ and z_2 defined as*



Premium	x^*	$\hat{a}$	premium
No insurance	0.257634	—	—
Exp. Value	0.249161	0.681818	0.0549664
Variance	0.200957	0.160305	0.0931698
Std. Deviation	0.235097	0.46577	0.0807122

Figure 14: Illustration of the results from Proposition 4.5 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.



Premium	x^*	$\hat{a}$	premium
No insurance	0.195074	—	—
Exp. Value	0.195074	1	0
Variance	0.193023	0.875	0.0129357
Std. Deviation	0.195074	1	0

Figure 15: Illustration of the results from Proposition 4.6 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

in (2.6). *Then,*

$$E(z_2(U,m)) = \int_0^m S_U(x)dx, \quad E(h_2(U,m)) = \int_m^1 S_U(x)dx$$

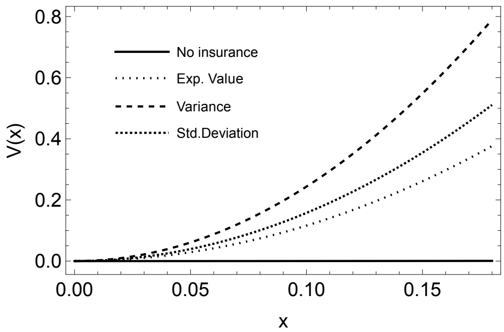
$$E(z_2^2(U,m)) = \int_0^m 2xS_U(x)dx, \quad E(h_2^2(U,m)) = \int_m^1 2(x-m)S_U(x)dx$$

$$E(z_2^{1/2}(U,m)) = \frac{1}{2}\int_0^m x^{-1/2}S_U(x)dx, \quad E(h_2^{1/2}(U,m)) = \int_m^1 1/2(x-m)^{-1/2}S_U(x)dx$$

$$E(z_2(U,l)) = \int_l^1 S_U(x)dx, \quad E(h_2(U,l)) = \int_0^l S_U(x)dx$$

$$E(z_2^2(U,l)) = \int_l^1 2(x-l)S_U(x)dx, \quad E(h_2^2(U,l)) = \int_0^l 2xS_U(x)dx$$

$$E(z_2^{1/2}(U,l)) = \int_l^1 \frac{1}{2}(x-m)^{-1/2}S_U(x)dx, \quad E(h_2^{1/2}(U,l)) = \frac{1}{2}\int_0^l x^{-1/2}S_U(x)dx$$



Premium	x^*	$\hat{a}$	premium
No insurance	6.97145	—	—
Exp. Value	0.293478	0	0.203478
Variance	0.202581	0	0.112581
Std. Deviation	0.251843	0	0.161843

Figure 16: Illustration of the results from Proposition 4.7 when the underlying risk is given by Scenario 1. Right: Value function for each optimal insurance contract. Left: Optimal investment thresholds, insurance contracts, and respective premia.

In the next proposition, we compute the expected value of the risk function $f_3(z(U)) = 1_{\{z(U)>q\}}$ for each of the insurance contracts considered.

Proposition B.2. *Let U be a random variable with support in the interval $(0, 1)$, the risk function $f_3(z(U)) = 1_{\{z(U)>q\}}$, and z_1, and z_2 defined as in (2.6). Then,*

$$E(f_3(z_1(U))) = \nu \left(a \int_{\frac{q}{a}}^{1} u f_U(u) du \right) 1_{\{a>q\}}$$

$$E(f_3(z_2(U;m))) = \nu \left(\int_{q}^{m} S_U(u) du + q S_U(q) \right) 1_{\{m>q\}} \quad \textit{and}$$

$$E(f_3(z_2(U;l))) = \nu \left(\int_{l+q}^{1} S_U(u) du + q S_U(l+q) \right) 1_{\{l<1-q\}}$$

Proof. To compute the expected value $E(f(z_1(U)))$ one should notice that

$$E(f(Z(U))) = \nu a \int_0^1 x 1_{\{ax>q\}} f_U(x) dx.$$

Taking into account that the support of the random variable U is the interval $(0, 1)$, then we obtain the expression in Proposition B.1.

The second expected value $E(f(z_2(U;m)))$ can be obtained noticing that

$$z_2(U;m) 1_{\{z_2(U;m)>q\}} = \begin{cases} 0, & m \le q \\ 0, & (U \le q) \wedge (m > q) \\ U, & (q < U < m) \wedge (m > q) \\ m, & (U \ge m) \wedge (m > q) \end{cases}.$$

The result in Proposition B.1 follows immediately noticing that for $m > q$

$$E(Z(u)1_{\{Z(U)>q\}}) = \int_q^M uf_U(u)\ du + \int_M^1 Mf_U(u)\ du = \int_q^M S_U(u)du + qS_U(q),$$

where the second equality follows as a result of the integration by parts.

Finally, to compute the expected value of $E(f(z_2(U;l)))$, we can observe that

$$z_2(U;l)1_{\{z_2(U;l)>q\}} = \begin{cases} 0, & (U \leq l+q) \wedge (l < 1-q) \\ (U-L), & (U > l+q) \wedge (l < 1-q) \\ 0, & l \geq 1-q \end{cases}.$$

The expected value in Proposition B.1 can be easily obtained noticing that for $l < 1-q$

$$\begin{aligned} E(Z(U)) = \int_{L+q}^1 (u-L)f_U(u)\ u = \Big[-uS_U(du)\Big]_{L+q}^1 + \int_{L+q}^1 S_U(u)du - LP(U > L+q) \\ = qS_U(L+q) + \int_{L+q}^1 S_U(u)du. \end{aligned}$$

∎

C Appendix: verification theorem

In this appendix, we present a verification result, which provides sufficient conditions for a solution of the HJB equation to be the value function of our problem. The proof of the next theorem follows the lines of Theorem 3.2 in [20], which we include for the sake of completeness.

Theorem C.1. *Let v be a differentiable function with absolutely continuous derivative such that v is a solution to HJB equation (3.1) and the process*

$$\left\{\int_0^{s\wedge t} \sigma X_u v'(X_u) dW_u\right\}_{s\geq 0} \quad \text{be a martingale.} \tag{C.1}$$

Then, for all $x > 0$,

1) *$v(x) \geq J(x,\tau,z)$, for all $\tau \in \mathcal{T}$ and $z \in \mathcal{Z}$;*
2) *if*

$$\lim_{t\to\infty} E_x\left[e^{-rt}v(X_t)1_{\{\tau^*>t\}}\right] = 0, \tag{C.2}$$

then $v(x) = J(x,\tau^*,z^*) = V(x)$, when $\tau^* = \inf\{t \geq 0 : v(X_t) = g(X_t)\}$.

Proof. Let v be a differentiable function with absolutely continuous derivative. Using the Itô-Tanaka formula (see [31]), we get that

$$\begin{aligned} E_x\left[e^{-r(t\wedge\tau)}g(X_{t\wedge\tau})\right] &= v(x) + E_x\left[e^{-r(t\wedge\tau)}\left(g(X_{t\wedge\tau}) - v(X_{t\wedge\tau})\right)\right] + E_x\left[\int_0^{t\wedge\tau} \sigma X_s v'(X_s)ds\right] \\ &= E_x\left[\int_0^{t\wedge\tau} \frac{\sigma^2 X_s^2}{2}v''(X_s) + \mu X_s v'(X_s) - rv(X_s)ds\right]. \end{aligned} \tag{C.3}$$

Since v satisfies the HJB equation (3.1) we get that

$$E_x\left[e^{-r(t\wedge\tau)}\left(g(X_{t\wedge\tau})-v(X_{t\wedge\tau})\right)\right]\leq 0,$$
$$E_x\left[\int_0^{t\wedge\tau}\frac{\sigma^2X_s^2}{2}v''(X_s)+\mu X_s v'(X_s)-rv(X_s)ds\right]\leq 0.$$

Condition (C.1) guarantees that

$$E_x\left[\int_0^{t\wedge\tau}\sigma X_s v'(X_s)ds\right]=0.$$

Then, we obtain that

$$E_x\left[e^{-r(t\wedge\tau)}g(X_{t\wedge\tau})\right]\leq v(x).$$

Using the strong Markov property and the dominated convergence theorem, one can easily see that

$$v(x)\geq\lim_{t\to 0}E_x\left[e^{-r(t\wedge\tau)}g(X_{t\wedge\tau})\right]=$$
$$\lim_{t\to 0}E\left[\int_{t\wedge\tau}^{\infty}e^{-rs}\left(X_s\left(1-\sum_{i=0}^{N_{s-(t\wedge\tau)}}\Big(U_i+f(z^*(U_i))+\alpha M(U_i-z^*(U_i))\Big)\right)-rK-\theta\right)dt\right]=$$
$$E\left[\int_{\tau}^{\infty}e^{-rs}\left(X_s\left(1-\sum_{i=0}^{N_{s-(\tau)}}\Big(U_i+f(z^*(U_i))+\alpha M(U_i-z^*(U_i))\Big)\right)-rK-\theta\right)dt\right].$$

Consequently, we have that

$$v(x)\geq J(x,\tau,z),$$

which proves 1).

To prove 2), we note that

$$E_x\left[e^{-r(t\wedge\tau^*)}v(X_{t\wedge\tau^*})\right]=E_x\left[e^{-r(t\wedge\tau^*)}\left(g(X_{\tau^*})1_{\{\tau^*\leq t\}}+v(X_t)1_{\{\tau^*>t\}}\right)\right],$$
$$E_x\left[\int_0^{t\wedge\tau^*}\frac{\sigma^2X_s^2}{2}v''(X_s)+\mu X_s v'(X_s)-rv(X_s)ds\right]=0.$$

combining these facts with condition (C.1) and Equation (C.3), we get that

$$E_x\left[e^{-r\tau^*}g(X_{\tau^*})1_{\{\tau^*\leq t\}}\right]=v(x)-E_x\left[e^{-rt}v(X_t)1_{\{\tau^*>t\}}\right].$$

As above, using the strong Markov property and the dominated convergence theorem, we obtain

$$J(x,\tau^*,z^*)=V(x),$$

when we assume that $t\to\infty$ as well as condition (C.2). ∎

D Appendix: Proofs

In this appendix, we present the proofs to Propositions 3.1, 3.2, and 4.1 to 4.7.

D.1 Proposition 3.1

The proof of this proposition follows the usual arguments for investment options [9]. To find the parameter A and the threshold x^*, we use the smooth-parting conditions, which allows us to obtain the following system of equations:

$$\begin{cases} A{x^*}^{\beta_1} &= g(x^*) \\ \beta_1 A{x^*}^{(\beta_1-1)} &= \dfrac{1}{r-\mu}\left(1-\dfrac{\lambda}{r-\mu}\left(u+E(f(z(U)))+\alpha M(U-z(U))\right)\right) \end{cases},$$

where $g(\cdot)=\tilde{g}(\cdot,z^*)$, with $\tilde{g}$ defined in (2.8). By solving the system of equations, we obtain the parameters A and x^*.

D.2 Proposition 3.2

It is a matter of calculations to verify that

$$\frac{\partial x^*}{\partial \lambda}=\frac{\left(K+\frac{\theta}{r}\right)\beta_1(r-\mu)^2\,(\beta_1-1)\,\left(u+E(f(z^*(U)))+\alpha M(h^*(U))\right)}{\Big((\beta_1-1)\left((r-\mu)-\lambda(u+E(f(z^*(U)))+\alpha M(h^*(U)))\right)\Big)^2}>0,$$

the last inequality following trivially from the assumptions in the model.

To compute the derivative of x^* regarding σ, we start by noting that

$$\frac{\partial \beta_i}{\partial \sigma}=(-1)^i\frac{\sigma\beta_i(\beta_i-1)}{\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\sigma^2 r}}.$$

Consequently, β_1 is decreasing with σ and β_2 is increasing with σ. We can write the derivative of x^* in order to σ as follows:

$$\frac{\partial x^*}{\partial \sigma}=\frac{-\frac{\partial \beta_i}{\partial \sigma}}{(\beta_1-1)^2}\times\frac{\left(K+\frac{\theta}{r}\right)(r-\mu)^2}{(r-\mu)-\lambda(u+E(f(z^*(U)))+\alpha M(h^*(U)))}>0.$$

D.3 Proposition 4.1

We look separately for each of the three contracts we are considering. Regarding the proportional contract, the optimization problem (4.1) reduces to

$$\min_{0\leq a\leq 1}\left\{\nu a^2\; u_2+\alpha(1-a)u\right\}.$$

Since we have a quadratic function in a, it is trivial that the minimum takes the form

$$\hat{a}=\begin{cases}\frac{\alpha u}{2\nu u_2}, & 2\nu u_2-\alpha u>0\\ 1, & 2\nu u_2-\alpha u\leq 0\end{cases}.$$

Regarding the contract with a deductible, using the expressions in Appendix B, one can easily see that the optimization problem (4.1) becomes

$$\min_{0\leq m\leq 1}\left\{2\nu\left(\int_0^m yS_U(y)dy\right)+\alpha\int_m^1 S_U(y)dy\right\}\equiv\min_{0\leq m\leq 1}\zeta(m).$$

Calculating the derivative

$$\frac{\partial\zeta}{\partial m}=S_U(m)(2m\nu-\alpha),$$

allows us to verify that the function ζ is decreasing in m while $m<\frac{\alpha}{2\nu}$, and it increases afterwards. Thus, the optimal m takes the form

$$\hat{m}=\begin{cases}\frac{\alpha}{2\nu}, & 2\nu-\alpha>0\\ 1, & 2\nu-\alpha\leq 0\end{cases}.$$

Finally, the optimization problem (4.1) for the contract with a limit becomes

$$\min_{0\leq l\leq 1}\left\{2\nu\left(\int_l^1 (y-l)S_U(y)dy\right)+\alpha\int_0^l S_U(y)dy\right\}\equiv\sup_{0<l<1}\zeta(l).$$

Calculating the derivative of ζ in l, we obtain that $\zeta'(l)=\alpha S_U(l)-2\nu\int_l^1 S_U(y)dy$. One can conclude that the monotony of ζ is dependent on the distribution of U. Consequently, we obtain the result in Proposition 4.1.

D.4 Proposition 4.2

We start by proving the result for the proportional contract. Taking into account Propositions B.1 and B.2, the minimization problem (4.1) can be written as $\min_{0\leq a\leq 1}\zeta(a)$ where

$$\zeta(a)=\nu\left(a\int_{\frac{q}{a}}^1 uf_U(u)du\right)1_{\{a>q\}}+\alpha(1-a)u.$$

The the function ζ is clearly decreasing whit a, when $a\leq q$. However, when $a\leq q$, the monotony of ζ depends on the shape of the density function of U. Thus, the optimal a is given by the minimum of $u+\nu a\int_{\frac{q}{a}}^1 uf_U(u)du+\alpha(1-a)u$ for $a\geq q$.

For a contract with a deductible, problem (4.1) can be written as $\min_{0\leq a\leq 1}\zeta(m)$ where the function ζ is defined as

$$\zeta(m)=\nu\left(\int_q^m S_U(y)dy+qS_U(q)\right)1_{\{m>q\}}+\alpha\int_m^1 S_U(y)dy.$$

Consequently, for $m<q$, $\frac{\partial\zeta}{\partial m}=-\alpha S_U(m)<0$, and for $m>q$, $\frac{\partial\zeta}{\partial m}=S_U(m)(\nu-\alpha)$. Therefore, there are two possible situations: if $\alpha>\nu$, function ζ is decreasing for all values of m; and if $\alpha<\nu$, function ζ decreases until $m=q$ and increases afterwards.

For the last case, the contract with a limit, the optimization problem (4.1) becomes $\min_{0\le l\le 1}\zeta(l)$, where

$$\zeta(l) = u + \alpha\int_0^l S_U(u)du + \nu\left(qS_U(l+q) + \int_{l+q}^1 S_U(u)du\right)1_{\{l<1-q\}}.$$

When $l < 1-q$, the derivative of function ζ is as follows:

$$\zeta'(l) = \alpha\ S_U(L) - \nu\ S_U(L+q) - \nu\ q\ f_U(L+q).$$

On the other hand, when $l > 1-q$, the derivative of function ζ is given by

$$\zeta'(l) = \alpha\ S_U(L) > 0.$$

Function ζ is continuous and thus its minimum is obtained as stated in Proposition 4.2.

D.5 Proposition 4.3

To obtain the result for the proportional contract we notice that the optimization problem (4.1) can be written as

$$\min_{0<a<1}\left\{\sqrt{a}\nu u_{1/2} + \alpha(1-a)u\right\} \equiv \min_{0\le a\le 1}\zeta(a).$$

Taking the derivative of ζ, we obtain that $\frac{\partial\zeta(a)}{\partial a} = \frac{\nu u_{1/2}}{2\sqrt{a}} - \alpha u$. Then, then the function ζ increases in $\left(0, \min\left(1, \left(\frac{\nu u_{1/2}}{2\alpha u}\right)^2\right)\right)$ and decreases in $\left(\min\left(1, \left(\frac{\nu u_{1/2}}{2\alpha u}\right)^2\right), 1\right)$. This means that ζ has an absolute maximum in the interval (0,1) and consequently, the minimum comes from the comparison between $\zeta(0)$ and $\zeta(1)$.

For contracts with a deductible, the optimization problem can be defined as $\sup_{0<m<1}\zeta(m)$ where

$$\zeta(m) = \frac{1}{2}\nu\int_0^m u^{\frac{-1}{2}} S_U(u)\ du + \alpha\int_m^1 S_U(u)du.$$

Taking into account that $\frac{\partial\zeta}{\partial m} = S_U(m)\left(\frac{\nu}{2\sqrt{m}} - \alpha\right)$, we use a similar argument to the previous one to obtain the optimal contract.

Finally, for contracts with a limit we have

$$\sup_{0<l<1}\zeta(L) \equiv \min_{0<l<1}\left\{\frac{1}{2}\nu\int_l^1 u^{\frac{-1}{2}} S_U(u)du + \alpha\int_0^l S_U(u)du\right\}.$$

Taking the derivative of ζ, we get $\frac{\partial\zeta}{\partial l} = S_U(l)\left(\alpha - \frac{\nu}{2\sqrt{l}}\right)$, which is decreasing for values of $l \le \frac{\nu^2}{4\alpha^2}$ and increasing otherwise. This proves the result.

D.6 Proposition 4.4

Part (i) is a consequence of Proposition 2.1. Regarding part (ii), one can see that (4.1) becomes

$$\min_{0<a<1}\Big\{\nu a u + \alpha(1-a)^2\sigma^2(U)\Big\}.$$

Calculating trivial derivatives allows us to observe that the optimal level of a is as in (ii). The last part can be proved following noticing that (4.1) can be written as

$$\min_{0<a<1}\Big\{\nu a u + \alpha(1-a)\sigma(U)\Big\}.$$

Since the function is linear in a we get our result.

D.7 Proposition 4.5

Part (i) was already proved in Proposition 4.1. To prove part (ii), one only needs to notice that (4.1) simplifies as

$$\min_{0<a<1}\Big\{\nu a^2 u_2 + \alpha(1-a)^2\sigma^2(U)\Big\}.$$

Taking the derivatives, one can see that the quadratic function as an absolute minimum a^*, as defined in the proposition, satisfying $0 < a^* < 1$, which proves this part of the proof.

Finally, to prove part (iii), we notice that (4.1) simplifies as

$$\min_{0<a<1}\Big\{\nu a^2 u_2 + \alpha(1-a)\sigma(U)\Big\}.$$

Since the minimum is attained at $a = \frac{\alpha\sigma_U}{2\nu u_2}$, which is not necessarily smaller than 1, we get a^* as defined in the proposition.

D.8 Proposition 4.6

Part (i) is proved in Proposition 4.2. The proof of part (ii) and (iii) follows the same steps because in all cases the function $M((1-a)U)$ is decreasing in a. Then the monotony for values of $a \in [q, 1]$ depends on the distribution of U.

D.9 Proposition 4.7

The proof is already done when we have the expected value premium principle. For the variance premium principle, one can easily see that $\hat{a}$ cannot be calculated analytically because the derivative of $\sqrt{a}\nu u_{1/2} + \alpha(1-a)^2\sigma^2(U)$ is given by

$$\frac{\nu\ u_{1/2}}{2\sqrt{a}} - 2\alpha\sigma^2(U)(1-a).$$

Finally, when the premium principle is the standard deviation, one can notice that

$$\arg\min_{a\in[0,1]}\left\{\sqrt{a}\nu u_{1/2} + \alpha(1-a)\sigma(U)\right\} \equiv \arg\min_{a\in[0,1]}\zeta(a).$$

Then, taking the derivative of $\zeta(a)$, we get $\zeta'(a) = \frac{\nu\ u_{1/2}}{2\sqrt{a}} - \alpha\sigma(U)$, which has a zero when $a = \left(\frac{\nu\ u_{1/2}}{2\alpha\sigma(U)}\right)^2$. This point is a maximizer if $\frac{\nu\ u_{1/2}}{2\alpha\sigma(U)} < 1$, otherwise the function ζ is always increasing. Consequently, the minimizer is $a = 0$ if $\nu \geq \frac{\alpha\sigma(U)}{u_{1/2}}$ or $a = 1$ otherwise.

References

[1] Daniel Aunon-Nerin and Paul Ehling. Why firms purchase property insurance. Journal of Financial Economics, 90(3):298–312, 2008.

[2] Florian Barth, Christian Eckert, Nadine Gatzert, and Hendrik Scholz. Spillover effects from the volkswagen emissions scandal: An analysis of stock and corporate bond markets. Schmalenbach Journal of Business Research, 74(1):37–76, 2022.

[3] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. Journal of political economy, 81(3):637–654, 1973.

[4] Karl Borch. The utility concept applied to the theory of insurance. ASTIN Bulletin: The Journal of the IAA, 1(5):245–255, 1961.

[5] Bernard Caillaud, Georges Dionne, and Bruno Jullien. Corporate insurance with optimal financial contracting. Economic Theory, 16:77–105, 2000.

[6] Tim S Campbell and William A Kracaw. Optimal managerial incentive contracts and the value of corporate insurance. Journal of Financial and Quantitative Analysis, 22(3):315–328, 1987.

[7] Jean-Paul Décamps, Thomas Mariotti, and Stéphane Villeneuve. Irreversible investment in alternative projects. Economic Theory, 28:425–448, 2006.

[8] Avinash K Dixit and Robert S Pindyck. The options approach to capital investment. In The economic impact of knowledge, pages 325–340. Routledge, 2009.

[9] Robert K Dixit and Robert S Pindyck. Investment Under Uncertainty. Princeton University Press, 1994.

[10] Christian Eckert. Corporate reputation and reputation risk: Definition and measurement from a (risk) management perspective. The journal of risk finance, 18(2):145–158, 2017.

[11] Franco Fiordelisi, Maria-Gaia Soana, and Paola Schwizer. Reputational losses and operational risk in banking. The European Journal of Finance, 20(2):105–124, 2014.

[12] Nadine Gatzert. The impact of corporate reputation and reputation damaging events on financial performance: Empirical evidence from the literature. European management journal, 33(6):485–499, 2015.

[13] Nadine Gatzert, Joan T Schmit, and Andreas Kolb. Assessing the risks of insuring reputation risk. Journal of Risk and Insurance, 83(3):641–679, 2016.

[14] Manuel Guerra, Peter Kort, Cláudia Nunes, and Carlos Oliveira. Hysteresis due to irreversible exit: Addressing the option to mothball. Journal of Economic Dynamics and Control, 92:69–83, 2018.

[15] Manuel Guerra, Cláudia Nunes, and Carlos Oliveira. Exit option for a class of profit functions. International Journal of Computer Mathematics, 94(11):2178–2193, 2017.

[16] Verena Hagspiel, Kuno JM Huisman, Peter M Kort, and Cláudia Nunes. How to escape a declining market: Capacity investment or exit? European Journal of Operational Research, 254(1):40–50, 2016.

[17] Robert E Hoyt and Ho Khang. On the demand for corporate property insurance. Journal of Risk and Insurance, pages 91–107, 2000.

[18] Rob Kaas, Marc Goovaerts, Jan Dhaene, and Michel Denuit. Modern actuarial risk theory: using R, volume 128. Springer Science & Business Media, 2008.

[19] Christopher R Knittel and Victor Stango. Celebrity endorsements, firm value, and reputation risk: Evidence from the tiger woods scandal. Management science, 60(1):21–37, 2014.

[20] Thomas S Knudsen, Bernhard Meister, and Mihail Zervos. Valuation of investments in real assets with implications for the stock prices. SIAM journal on control and optimization, 36(6):2082–2102, 1998.

[21] Peter Kort, Maria Lavrutich, Cláudia Nunes, and Carlos Oliveira. Preventing environmental disasters in investment under uncertainty. Environmental and Resource Economics, pages 1–22, 2021.

[22] Peter Kort, Maria Lavrutich, Cláudia Nunes, and Carlos Oliveira. Preventive investment, malfunctions and liability. International Journal of Production Economics, page 108930, 2023.

[23] Simone Krummaker. Firm's demand for insurance: An explorative approach. Risk Management and Insurance Review, 22(3):279–301, 2019.

[24] Brian GM Main. Corporate insurance purchases and taxes. Journal of Risk and Insurance, pages 197–223, 1983.

[25] David Mayers and Clifford W Smith Jr. On the corporate demand for insurance. In Foundations of Insurance Economics: Readings in Economics and Finance, pages 190–205. Springer, 1982.

[26] David Mayers and Clifford W Smith Jr. Corporate insurance and the underinvestment problem. Journal of Risk and Insurance, pages 45–54, 1987.

[27] Stewart C Myers. Interactions of corporate financing and investment decisions-implications for capital budgeting. The Journal of finance, 29(1):1–25, 1974.

[28] Bernt Oksendal. Stochastic differential equations: an introduction with applications. Springer Science & Business Media, 2013.

[29] Jason Perry and Patrick De Fontnouvelle. Measuring reputational risk: The market reaction to operational loss announcements. Available at SSRN 861364, Federal Reserve Bank of Boston, 2005.

[30] Goran Peskir and Albert Shiryaev. Optimal stopping and free-boundary problems. Springer, 2006.

[31] Daniel Revuz and Marc Yor. Continuous martingales and Brownian motion, volume 293. Springer Science & Business Media, 2013.

[32] Matteo Tonello. Reputation risk: A corporate governance perspective. The Conference Board Research Report No R-1412-07-WG, 2007.

[33] Ingo Walter. The value of reputational capital and risk in banking and finance. International Journal of Banking, Accounting and Finance, 5(1-2):205–219, 2013.

[34] Mihail Zervos, Carlos Oliveira, and Kate Duckworth. An investment model with switching costs and the option to abandon. Mathematical Methods of Operations Research, 88:417–443, 2018.