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Zaichuan Du

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Rua Miguel Lúpi 20,
1249-078 Lisboa,
Portugal

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REM – Research in Economics and Mathematics

Rua Miguel Lupi, 20
1249-078 LISBOA
Portugal

Telephone: +351 - 213 925 912

E-mail: rem@iseg.ulisboa.pt

<https://rem.rc.iseg.ulisboa.pt/>



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Determinacy and Analytical Tail Solutions in the Sequence Space

Zaichuan Du*

ISEG, University of Lisbon

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Abstract

We develop a continuous-time geometric formulation of local equilibrium in sequence space. Household optimality, distributional transport, and market clearing define a common equilibrium relation whose linearization is an operator on admissible paths. Its kernel measures first-order indeterminacy; its cokernel identifies restrictions on shocks required for existence. Under an exponential dichotomy and a closed-range condition, we reduce both questions to the map from stable initial traces to predetermined data. The same stable–unstable decomposition yields an exact analytic boundary condition for an infinite stationary tail. Solving a finite prefix with this condition is equivalent to solving the original half-line problem and introduces no time-horizon truncation error.

JEL codes: C61, C63, E21

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*Email: du@zaichuan.net. The appendix and replication code are available upon request by email.

1 Introduction

Sequence-space methods express equilibrium as a relation among entire paths of aggregate variables. This is especially useful when individual heterogeneity creates a large distributional state but equilibrium depends on a comparatively small collection of prices and aggregates (Auclert et al., 2021). The sequence-space Jacobian (SSJ) is the derivative of this path relation around a reference equilibrium. However, its definition is intrinsically infinite-dimensional. Even a model with finitely many aggregate variables generally requires paths on an unbounded time interval.

Three questions therefore belong to the same problem. Does an equilibrium response exist for a given disturbance? Is that response unique within the economically admissible class? How can one compute it without replacing the distant future by an arbitrary terminal restriction? Standard finite-horizon calculations can obscure the relation among these questions. An invertible truncated matrix does not by itself establish well-posedness of the half-line equilibrium, and a small numerical residual on a finite interval does not ensure a correct continuation after that interval.

Auclert et al. (2025) address these questions through the quasi-Toeplitz structure of stationary discrete-time Jacobians. Their paper links infinite-dimensional operator theory, determinacy diagnostics, compact corrections, and iterative solution. We take a continuous-time route. Our starting point is the coupled Hamilton–Jacobi–Bellman (HJB), Kolmogorov forward (KFE), and market-clearing system, including initial conditions and admissibility at infinity. Its geometry makes clear which parts of the equilibrium problem concern transport, which concern feedback, and which concern boundary data.

We define general equilibrium as a fiber product of sectoral solution relations with a common market-consistency relation. This definition is independent of the choice of prices used to parameterize equilibrium. The derivative of a particular residual representation, however, is coordinate dependent. What survives equivalent representations is the tangent solution relation and its obstruction space. Under a closed-range assumption, the kernel records homogeneous admissible perturbations, while the adjoint kernel records compatibility restrictions on forcing. This distinction separates uniqueness from existence and prevents mass conservation from being mistaken for a determinacy theorem.

The first main result gives this geometry a concrete form. After eliminating regular algebraic constraints, suppose the linearized dynamics admit an exponential dichotomy. Admissible homogeneous paths are indexed by their stable initial traces. The restriction of the initial-data map to those traces then determines both defects: its kernel measures indeterminacy, and its cokernel measures missing compatibility conditions. Thus determinacy requires more than a count of stable modes. The stable solution space must intersect the initial-data constraints with the correct orientation. This is a bounded Hilbert-space realization, which includes every fixed finite spatial discretization, and specify what is additionally needed for an unbounded PDE realization.

The second main result uses the same geometry to remove an infinite tail exactly. Beyond a matching date T , suppose the linearized coefficients are stationary. The unstable component at

T is determined by all future forcing. The stable component remains free for matching to the early solution. This condition defines an affine admissible-trace space at T . Coupling the finite prefix to that space is equivalent to the original infinite-horizon boundary-value problem. No assumption that the response vanishes at T is made. For a stationary steady-state linearization, stationarity holds at every date. T is only a computational interface. For a genuinely time-varying model, the same exact statement requires an exact continuation operator. Replacing coefficients that merely converge to a limit by that limit creates an approximation error.

We use a discrete exterior calculus (DEC) scheme to realize the continuous transport geometry on a spatial mesh, as developed in our companion paper Du (2026). A common incidence structure yields constant-preserving backward dynamics, mass-preserving forward dynamics, and mass-zero policy derivatives. Those properties protect the domain on which closure and tail resolvents are computed. For the basic household feedback block, we leverage the fake news operator to construct projection actions by integrating the price-response correction to a known household split.

Relation to the literature. The SSJ approach of Auclert et al. (2021) provides the path-space computational foundation. Bilal and Goyal (2026) derive policy responses analytically in continuous time and use variable time grids. Their analytic policy differentiation can supply the household block in our construction, while our focus is the admissible general-equilibrium closure and the infinite tail. Continuous-time HJB–KFE models are developed by Achdou et al. (2022). DEC supplies compatible discrete differential operators (Desbrun et al., 2005). In the companion paper Du (2026), we develop a geometric DEC scheme to solve HACT. The dichotomy argument belongs to the established theory of half-line boundary-value problems. Morris (2011) gives a related nonlinear Fredholm treatment. Our contribution is that we keep the geometric structure in the generator that produces the quasi-Toeplitz input-output operator. The geometry that proves determinacy is also the geometry that supplies the exact analytic tail. DEC provides a conservative spatial realization that preserves the distributional tangent space on which the closed-loop generator and its tail operators are defined. We combine the analytical household responses of Bilal and Goyal (2026) with the admissible stable/unstable splitting of the complete equilibrium realization. The resulting terminal feedback enters the household Jacobian action itself, retaining aggregate paths as the unknowns of the SSJ solve.

Section 2 prepares the household operators, admissible paths, and equilibrium geometry. Section 3 establishes the existence and uniqueness criteria and their sequence-space representation. Section 4 constructs the finite-grid spectral split and uses it to test determinacy, compute the calibrated Aiyagari response, and complete finite-prefix SSJ calculations with exact continuation. Section 5 concludes. Appendices provide proofs and computational derivations. Appendix D develops the household Jacobian, terminal-feedback algorithm, and analytical verification of the tail.

2 Household operators and equilibrium geometry

The equilibrium unknowns are paths of prices and quantities. To describe their economic restrictions, we retain the household value and distribution as internal paths. Prices affect decisions, decisions move households across individual states, and aggregation feeds those movements back into market clearing. This section defines the operators in that loop and the admissible path spaces on which the equilibrium derivative $\mathcal{D}_{\text{eq}}$ acts. Section 3 uses their closed equilibrium realization to study stable traces. Section 4 constructs those traces and uses the resulting split for equilibrium computation and SSJ continuation. Appendix D derives the household Jacobian and the elimination of internal paths with that continuation imposed.

The geometry records the type of each economic object. A value function is a scalar function of individual states. A saving policy induces a velocity and a probability flux. The distribution is a measure, possibly including boundary atoms. Smooth densities can be represented by top-degree forms, while a DEC realization records values at nodes and transport across edges. Aggregates are pairings between observables and distributions. These identifications organize the backward–forward duality and population accounting. The equilibrium complex introduced in Section 3 concerns the complete linearized residual, including its initial and boundary conditions.

2.1 Levels, steady states, and first-order responses

A variable without a perturbation marker denotes a level in the nonlinear economy. Thus $V(t, a, i)$ is the value function, $\mu(t)$ the household distribution, $p(t)$ a vector of prices, and $z(t)$ a vector of aggregate exogenous variables. A bar denotes the stationary reference equilibrium. Consider an experiment indexed by its size ε , such as a small productivity disturbance. We write

$$\begin{aligned} V^\varepsilon &= \bar{V} + \varepsilon v + o(\varepsilon), & \mu^\varepsilon &= \bar{\mu} + \varepsilon g + o(\varepsilon), \\ p^\varepsilon &= \bar{p} + \varepsilon \delta p + o(\varepsilon), & z^\varepsilon &= \bar{z} + \varepsilon \delta z + o(\varepsilon). \end{aligned} \tag{2.1}$$

Here $v = \delta V$ and $g = \delta \mu$ are convenient shortcuts for first-order deviations. Lower case by itself does not mean a deviation. For example, p is a level and δp its derivative with respect to ε . The actual price change is approximately $\varepsilon \delta p$. All deviations are in levels unless a log deviation is explicitly specified. A percentage response is obtained by dividing by the relevant steady-state level. A signed distribution change g is not a probability distribution.

The perturbation derivative δ should also be distinguished from a time derivative ∂_t , a wealth derivative ∂_a , and an operator derivative $D\mathcal{A}$. In particular, $D\mathcal{A}(\bar{p})[\delta p]$ means the first-order change in an entire output path when the entire input price path changes in direction δp . A hat is reserved for the Laplace transform in Section 3. It does not denote a first-order deviation. We use z_i for an idiosyncratic productivity state in the Aiyagari example. that state is distinct from the vector of aggregate shocks $Z(t)$.

Table 1: Notation for the economic objects and their linear responses

Economic object	Level	First-order response	Interpretation
Value function	V , steady state $\bar{V}$	$v = \delta V$	Change in household continuation value
Distribution	μ , steady state $\bar{\mu}$	$g = \delta \mu$	Redistribution of a fixed population
Prices	p , steady state $\bar{p}$	δp	For example, $(\delta r, \delta w)$
Consumption rule	c , steady state $\bar{c}$	δc	Response at a given individual state
Aggregate output	Y , steady state $\bar{Y}$	δY	Response after aggregation
Exogenous inputs	z , steady state $\bar{z}$	δz	Specified shock paths

2.2 Path spaces and linear response operators

An SSJ takes a small input path and returns the associated small output path. For a household aggregate map $Y = \mathcal{A}(p)$, its derivative is

$$\delta Y = \mathcal{J} \delta p, \quad \mathcal{J} = D\mathcal{A}(\bar{p}), \quad \delta Y(t) = \int_0^\infty J(t, \tau) \delta p(\tau) d\tau, \quad (2.2)$$

with a direct multiplication term included when necessary. The two dates have different roles: τ is the date of the price change, and t is the date at which we measure the aggregate response. Anticipation permits $t < \tau$. A kernel column for a dated impulse is a convenient distributional notation. Actual admissible inputs can be ordinary square-integrable paths or smooth approximations to that impulse.

We need a class of paths in which “returning to equilibrium” has a precise meaning. Our baseline is the space of square-integrable responses. For a vector or function-valued path x , define

$$L_\eta^2(E) = \left\{ x : \int_0^\infty e^{-2\eta t} \|x(t)\|_E^2 dt < \infty \right\}, \quad \|x\|_\eta^2 = \int_0^\infty e^{-2\eta t} \|x(t)\|_E^2 dt. \quad (2.3)$$

The baseline is $\eta = 0$. For aggregate variables, E can simply be Euclidean space with a chosen scaling of its components. For values or distributions, the norm also aggregates across individual states. A Hilbert space is a setting in which these norms arise from an inner product, so orthogonality and least-squares projections are available. These technical choices make the kernel and cokernel calculations in Section 3 well defined.

A positive η allows some growing responses and changes admissibility. Permanent shocks are better handled by specifying the new steady state or the relevant center dynamics explicitly. When equations contain $\dot{x}$, the solution space must also control its derivative. H^1 is the space in which both x and $\dot{x}$ are square integrable. Its initial trace is simply the well-defined value $x(0)$. For a continuous-state PDE, a suitable graph norm additionally controls the spatial generator. These details matter because an equilibrium response must satisfy both the dynamic equations and the given initial distribution.

2.3 Household decisions and the evolution of the distribution

Let Ω be the individual state space. The backward Markov generator $\mathcal{L}_{\alpha,p}$ describes how a function of individual states changes when households follow policy α at prices p . A generator is the continuous-time counterpart of a transition matrix. If $\varphi(a, i)$ is an observable, such as assets or consumption, $e^{t\bar{\mathcal{L}}}\varphi$ is its expected value after t units of time, conditional on the current state and steady-state behavior. The forward operator $e^{t\bar{\mathcal{L}}^*}$ instead evolves a distribution.

These are two descriptions of the same household transitions. The pairing is the aggregate

$$\langle \varphi, \mu \rangle = \sum_i \int \varphi(a, i) \mu_i(da).$$

Computing that aggregate by moving households forward or by moving the observable backward must give the same answer:

$$\langle e^{t\bar{\mathcal{L}}}\varphi, \mu \rangle = \langle \varphi, e^{t\bar{\mathcal{L}}^*}\mu \rangle, \quad \langle \mathcal{L}\varphi, \mu \rangle = \langle \varphi, \mathcal{L}^*\mu \rangle, \quad \mathcal{L}\mathbf{1} = 0. \quad (2.4)$$

This is the economic meaning of adjointness. The last identity says that expected population size does not change. Boundary conditions are included in the generator. Borrowing constraints or boundary masses must be treated with the corresponding weak flux conditions.

In levels, household optimality, distribution dynamics, and market clearing take the form

$$\begin{aligned} 0 &= \rho V - \partial_t V - \mathcal{H}(V, p, z), \\ 0 &= \partial_t \mu - \mathcal{L}_{\alpha(V,p,z),p,z}^* \mu, \\ 0 &= \mathcal{C}(V, \mu, p, z), \quad \mu(0) = \mu_0. \end{aligned} \quad (2.5)$$

The optimized Hamiltonian $\mathcal{H}$ collects current utility and expected continuation value. $\mathcal{C}$ collects equilibrium restrictions. The HJB determines optimal behavior given prices, while the KFE determines the distribution given behavior. Their joint solution becomes a general equilibrium only when $\mathcal{C} = 0$. Predetermined aggregate states can be added to the distributional state. The formulation concerns perfect-foresight transitions and their first-order impulse responses, not a nonlinear common-noise master equation.

Because total population is normalized to one, $\langle \mathbf{1}, \mu^\varepsilon \rangle = 1$ for every admissible experiment. Differentiating this accounting identity gives

$$\mathcal{M}_0 = \{g : \langle \mathbf{1}, g \rangle = 0\}. \quad (2.6)$$

The space $\mathcal{M}_0$ is the distributional tangent space: a first-order response can move households between states but cannot create households. It is an economic accounting restriction, not an additional equilibrium condition. If the reference distribution is on the boundary of the set of feasible distributions, nonlinear perturbations can be constrained to a cone. Local nonlinear results then require a suitable differentiable chart or constrained argument.

Differentiating the KFE separates the persistence of an existing distribution change from the effect of changed behavior:

$$\dot{g} = \bar{\mathcal{L}}^* g + (D\mathcal{L}[\delta\alpha, \delta p, \delta z])^* \bar{\mu}. \quad (2.7)$$

The first term propagates households using steady-state transitions. The second term real-locates the steady-state population because saving, adjustment, or transition policies have changed. In a regular one-asset density representation, its saving component is $-\partial_a(\bar{\mu} \delta s)$. This is the divergence of the additional asset flow. Its integral is zero when the full boundary flux is accounted for. The same conclusion follows directly by differentiating $\mathcal{L}\mathbf{1} = 0$. Removing the mass direction is necessary for distributional response calculations, but a spectral gap on $\mathcal{M}_0$ still requires a separate mixing assumption.

Linearized household operators. Fix the steady state and temporarily take aggregate prices as exogenous. The analytically differentiated household block can be organized as

$$\begin{aligned} -\dot{v} &= Hv + B\delta p, \\ \dot{g} &= Ag + Cv + D\delta p, \quad g(0) = 0, \\ \delta Y &= Kg + Nv + R\delta p. \end{aligned} \tag{2.8}$$

Here all lowercase paths on the left are deviations as defined above. The coefficients are evaluated at the steady state. The operator $A = \bar{\mathcal{L}}^*|_{\mathcal{M}_0}$ propagates distribution changes. In the simplest envelope linearization, $H = \bar{\mathcal{L}} - \rho I$ propagates discounted value news. The map B converts a price change into a change in the HJB payoff. The map C translates the resulting policy response into a distributional source. D captures direct effects of prices on individual-state transitions. Both have mass-zero outputs.

The last equation is aggregation. For asset supply, Kg is the change in assets due to redistribution and usually $N = R = 0$. For consumption, there is also a response of consumption at a fixed household state: Nv captures the value-induced policy effect and $R\delta p$ any direct effect. The letter K in this generic block is an aggregation operator. Aggregate capital in the Aiyagari example will be explicitly identified as the variable $K(t)$.

This representation accommodates analytical policy differentiation of the type developed by Bilal and Goyal (2026). For example, differentiating the consumption policy on a fixed binding set gives the linear map $\delta c = C_c v + D_c \delta p$. Here C_c maps a value-function perturbation into a consumption response, while D_c maps current price changes into consumption holding the value-function input fixed. In the baseline model, the interior first-order condition $U'(c) = \partial_a V$ gives $C_c v = \partial_a v / U''(\bar{c})$ and $D_c = 0$. Prices still affect interior consumption through v . At a borrowing boundary that remains binding, consumption instead tracks current disposable income. Appendix B.3.4 states the corresponding policy derivatives.

These policy maps determine the distributional maps C and D . Writing $b\delta p = a\delta r + y_i\delta w$ for the income response at a fixed household state, the saving response is $\delta s = -C_c v + (b - D_c)\delta p$. The transport-source map $\mathcal{S}\chi = -\partial_a(\bar{\mu}\chi)$ converts an admissible saving-drift perturbation χ into the instantaneous redistribution of the steady-state population. Thus $C = -\mathcal{S}C_c$ and $D = \mathcal{S}(b - D_c)$: the consumption response enters saving with a minus sign, and the direct saving response is income minus consumption. Unlike C_c and D_c , whose outputs are individual policy responses, C and D produce mass-zero sources for the distribution equation. Appendix B.3.5 defines $\mathcal{S}$ weakly, explains its flux interpretation, and derives these identities with the boundary included.

Instead of repeatedly solving the nonlinear household problem for perturbed price sequences, one solves its linear response equation. Under exponential stability of H and A , the resulting paths are equivalent to the analytical result in Bilal and Goyal (2026)

$$\begin{aligned} v(t) &= \int_t^\infty e^{H(\tau-t)} B \delta p(\tau) d\tau, \\ g(t) &= \int_0^t e^{A(t-r)} \{Cv(r) + D\delta p(r)\} dr. \end{aligned} \tag{2.9}$$

The first line integrates anticipated future price effects backward to today's value. The second integrates past saving responses forward to today's distribution. These are variation-of-constants formulas, often called Duhamel formulas. In the bounded realization used here, all coupling maps are bounded. PDE versions need the corresponding domain and input/output regularity conditions.

We denote these backward and forward solution operations by $\mathcal{V}$ and $\mathcal{G}$: $v = \mathcal{V}\delta p$ and $g = \mathcal{G}(Cv + D\delta p)$ for zero initial distribution deviation. A prescribed g_0 adds $e^{At}g_0$ to the forward response. The maps H, A, B, C, D , together with the aggregation maps K, N, R , are the building blocks of the full derivative $\mathcal{D}_{\text{eq}}$. Appendix D.1 composes them into aggregate Jacobians. Their local evolution equations also give a realization of those path operators through which the stable-subspace and continuation calculations can be made.

2.4 Closing the household block with the rest of the economy

The household block describes decisions and distributional responses given prices. General equilibrium requires those same prices to be consistent with firms, fiscal and monetary policy, and market clearing. In a one-asset production economy, a trial capital path determines factor prices. Households respond to those prices. Their asset holdings must reproduce the original capital path. This feedback is what we call *closure*. Geometry gives a precise description of the intersection of these mutually consistent sectoral choices.

For sector i , let $\mathcal{S}_i(z)$ collect all paths satisfying that sector's equations and boundary conditions for a given exogenous path. The interface map π_i extracts the prices and quantities that the sector exchanges with others. Let $\mathcal{K}(z)$ impose agreement on those interfaces: the same wage is paid and received, assets supplied equal assets demanded, and the relevant budgets balance. The complete equilibrium relation is

$$\mathcal{M}_{\text{GE}}(z) = \left(\prod_i \mathcal{S}_i(z) \right) \times_{\prod_i \mathcal{P}_i} \mathcal{K}(z). \tag{2.10}$$

The term *fiber product* means that we retain precisely the sectoral solutions that agree on their shared economic variables. It does not introduce a new equilibrium condition. Its advantage is to define the complete equilibrium before choosing which prices or aggregates a numerical solver will treat as unknowns. It is a smooth manifold only under appropriate regularity conditions.

For example, solving for the capital path or for the interest-rate path can be two parameterizations of the same equilibrium, provided the conversion is locally invertible. An evaluation graph for partial-equilibrium blocks depends on that choice. The underlying household–firm

consistency relation does not. This distinction is useful when both households and firms are heterogeneous and their distributions feed back into each other's decisions.

Let x collect the endogenous *level* paths and let $\mathcal{F}(x, z)$ stack their equilibrium residuals. Its first-order approximation is

$$\mathcal{F}(x, z) = 0, \quad \mathcal{D}_{\text{eq}} = D_x \mathcal{F}(\bar{x}, \bar{z}) : \mathcal{X} \rightarrow \mathcal{Y}, \quad \mathcal{D}_{\text{eq}} \delta x = \mathcal{B} \delta z, \quad \mathcal{B} = -D_z \mathcal{F}(\bar{x}, \bar{z}). \quad (2.11)$$

The residual space $\mathcal{Y}$ includes the predetermined initial data as well as the dynamic and market-clearing equations. Known redundancies, such as one Walras-law equation, must be removed before counting existence restrictions. The linearized household operators in (2.8) supply its optimality and distribution blocks. Eliminating internally determined household paths gives an aggregate SSJ closure. Section 3 proves that valid elimination preserves the uniqueness and existence tests. Section 4.6 states the corresponding exact-tail equivalence, and Appendix D develops the Jacobian computation.

Changing coordinates gives $\tilde{\mathcal{D}} = U \mathcal{D} V$, with invertible transformations U, V . Although the matrix entries change, the set of homogeneous equilibrium responses and the number of independent compatibility restrictions do not. This is the precise sense in which the closure geometry is independent of the chosen economic coordinates.

Evaluation graphs and equilibrium closure. An evaluation graph orders the sectoral calculations conditional on proposed aggregate paths. Its nodes may include household decisions, distributional transport, firms, and policy rules. Market clearing closes that graph by requiring agreement on the interfaces. The aggregate unknowns are a locally independent set of paths sufficient to specify those interfaces. Changing that set changes the evaluation graph and the numerical representation, while valid elimination preserves the complete equilibrium relation. Determinacy therefore concerns $\mathcal{D}_{\text{eq}}$, including all boundary data, rather than a particular ordering of partial-equilibrium calculations.

3 Determinacy, existence and geometric closure

Section 2 defines the household operators and their equilibrium coupling. We now ask whether all sectors can agree on one response to the same shock. Two separate economic questions arise. *Uniqueness* asks whether a second admissible response could satisfy the same equations with the same inherited states and shocks. *Existence* asks whether those inherited states and shocks can be accommodated by any admissible response. The first concerns movements the equilibrium equations do not pin down. The second concerns restrictions endogenous adjustment cannot satisfy. The kernel and cokernel are the linear-algebraic objects that measure these two failures.

We first state the general path-space criterion, then express it through stable equilibrium dynamics and relate it to the aggregate sequence-space operator. Section 4 constructs the stable split from analytical household responses and implements the criterion in an Aiyagari economy. Throughout, $u(t)$ denotes a stacked vector of $\{\text{first-order deviations}\}$. It is not a vector of level variables.

3.1 The path operator, its kernel, and its obstruction space

Let $\delta x = (v, g, \delta p)$ collect the value, distribution, and price responses. Stack the linearized HJB, KFE, and market-clearing residuals in that order:

$$\mathcal{D}_{\text{eq}} \begin{pmatrix} v \\ g \\ \delta p \end{pmatrix} = \begin{pmatrix} -\partial_t - H & -B_{vg} & -B_{vp} \\ -B_{gv} & \partial_t - A & -B_{gp} \\ C_v & C_g & C_p \end{pmatrix} \begin{pmatrix} v \\ g \\ \delta p \end{pmatrix}, \quad (3.1)$$

The initial-condition equations are appended to this displayed interior operator. The domain includes the relevant spatial boundary conditions and admissibility at infinity. Here B_{vp} is the effect of prices on the household optimality equation, B_{gv} the policy-induced redistribution map, and B_{gp} the direct price-induced redistribution map. The last-row coefficients differentiate the market residuals. The first subscript identifies the affected equation and the second the input variable. In the basic household problem $B_{vg} = 0$: an atomistic household takes prices as given, so the distribution affects it through prices rather than as an additional argument of its decision problem. Richer specifications can produce a direct distribution dependence.

To say that δx lies in the kernel is to set all these residuals, all shock inputs, and all initial-data deviations to zero and still obtain a nonzero path. Adding that path to an existing response leaves the economic experiment unchanged. Conversely, a left-null direction is a set of weights on the equations whose weighted residual cannot be changed by any endogenous path. If the exogenous shock generates a nonzero residual in that direction, the equilibrium equations are incompatible. These are the infinite-path analogues of the right and left nullspaces of a matrix.

Assumption 1 (Regular linear closure). *The chosen path and residual spaces $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces. The differential expressions in (3.1), including their traces, define a bounded map $\mathcal{D}_{\text{eq}} : \mathcal{X} \rightarrow \mathcal{Y}$ when $\mathcal{X}$ carries its graph norm. The range is closed. The forcing map $\mathcal{B} : \mathcal{Z} \rightarrow \mathcal{Y}$ is bounded. Independent normalization and accounting identities have been dealt with in defining these spaces.*

Definition 1 (Closure defects and admissible disturbances). *The linearized indeterminacy space and obstruction space are*

$$\mathcal{H}_{\text{eq}}^0 = \ker \mathcal{D}_{\text{eq}}, \quad \mathcal{H}_{\text{eq}}^1 = \mathcal{Y} / \text{Ran } \mathcal{D}_{\text{eq}} \simeq \ker \mathcal{D}_{\text{eq}}^*. \quad (3.2)$$

The economically admissible disturbance set is

$$\mathcal{Z}_{\text{adm}} = \{\delta z \in \mathcal{Z} : \mathcal{B}\delta z \in \text{Ran } \mathcal{D}_{\text{eq}}\}. \quad (3.3)$$

The adjoint in this definition is the Hilbert-space adjoint of the complete operator, including the boundary component. $\text{Ran } \mathcal{D}_{\text{eq}} = \{\mathcal{D}_{\text{eq}}\delta x : \delta x \in X\}$ is the set of sequence space data that some admissible response path can satisfy, where δx collects the entire paths of value-function, distribution, and price responses. The space X imposes the required spatial boundary conditions and infinite-horizon admissibility. $\mathcal{B}\delta z$ specifies what exogenous disturbance requires the equations to accommodate. $\mathcal{B}\delta z \in \text{Ran } \mathcal{D}_{\text{eq}}$ means that at least one admissible response δx solves the equilibrium equations. It asks whether adjustments in household choices, the distribution, and prices can reconcile the disturbance with all equilibrium conditions.

Theorem 1 (Linear geometric closure). *Under Assumption 1, the following statements hold.*

1. *For a given δz , an equilibrium perturbation exists if and only if $\langle \psi, \mathcal{B}\delta z \rangle_{\mathcal{Y}} = 0$ for every $\psi \in \ker \mathcal{D}_{\text{eq}}^*$.*
2. *Whenever a solution x_p exists, the full solution set is $x_p + \mathcal{H}_{\text{eq}}^0$. Hence uniqueness of existing solutions is equivalent to $\mathcal{H}_{\text{eq}}^0 = \{0\}$.*
3. *There is a unique continuously dependent solution for every residual forcing in $\mathcal{Y}$ if and only if both defect spaces vanish. In that case $\mathcal{D}_{\text{eq}}^{-1}$ is bounded.*
4. *If $\mathcal{D}_{\text{eq}}$ is Fredholm, its index is $\dim \mathcal{H}_{\text{eq}}^0 - \dim \mathcal{H}_{\text{eq}}^1$. An index of zero does not imply that either defect vanishes.*

Proof. The closed-range theorem gives $\text{Ran } \mathcal{D}_{\text{eq}} = (\ker \mathcal{D}_{\text{eq}}^*)^\perp$, proving the first assertion. Subtracting two solutions proves the second. A bounded bijection between Banach spaces has a bounded inverse, proving the third. The fourth is the definition of the Fredholm index. Equal nonzero kernel and cokernel dimensions demonstrate why zero index is insufficient. $\square$

$\mathcal{H}^0$ counts unpinned equilibrium movements and $\mathcal{H}^1$ counts independent compatibility restrictions. A Fredholm operator is one with closed range and finitely many such directions. Its index reports their difference, so a zero index can conceal equally many failures on each side. Geometrically, the two spaces are associated with the two-term deformation complex

$$0 \longrightarrow \mathcal{X} \xrightarrow{\mathcal{D}_{\text{eq}}} \mathcal{Y} \longrightarrow 0. \quad (3.4)$$

Equilibrium closure means exactness at both nonzero terms. This is an analytic statement about economic dynamics and boundary conditions.

3.2 Saddle-path selection and a constructive determinacy criterion

The usual saddle-path logic in macroeconomics provides a constructive way to evaluate the abstract criterion. Predetermined states, such as the household distribution, are inherited at date zero. Jump variables, such as values and some asset prices, adjust to make the subsequent response admissible. Eliminate static variables, the remaining first-order system is

$$\dot{u} = \mathbf{A}u + f, \quad \Gamma u(0) = d. \quad (3.5)$$

The vector u includes forward-looking variables and mass-zero distribution states. The boundary space $\mathbf{D}$ contains predetermined data, and $\Gamma : E \rightarrow \mathbf{D}$ is bounded.

The operator $\mathbf{A}$ is obtained by solving the static equations in $\mathcal{D}_{\text{eq}}$ and rearranging its differential equations. To make the construction explicit, denote the three interior forcing components in (3.1) by (b_v, b_g, b_p) , in HJB, KFE, and market-clearing order. They are the corresponding components of $\mathcal{B}\delta z$. Inherited initial data are imposed separately. Suppose the instantaneous price block C_p is boundedly invertible. The market equation can then be solved at each date:

$$C_v v + C_g g + C_p \delta p = b_p, \quad \delta p = -C_p^{-1} C_v v - C_p^{-1} C_g g + C_p^{-1} b_p. \quad (3.5a)$$

Substitute this expression into the first two rows of (3.1), and multiply the HJB row by -1 to isolate $\dot{v}$. With $u = (v, g)$, the resulting generator and forcing in (3.5) are

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} -H + B_{vp}C_p^{-1}C_v & -B_{vg} + B_{vp}C_p^{-1}C_g \\ B_{gv} - B_{gp}C_p^{-1}C_v & A - B_{gp}C_p^{-1}C_g \end{pmatrix}, \\ f &= \begin{pmatrix} -b_v - B_{vp}C_p^{-1}b_p \\ b_g + B_{gp}C_p^{-1}b_p \end{pmatrix}. \end{aligned} \quad (3.5b)$$

The products involving C_p^{-1} are the price feedback implied by market clearing. They transmit changes in values and the distribution back into household optimality and distribution dynamics.

For (2.8), take $B_{vg} = 0$, $B_{vp} = B$, $B_{gv} = C$, and $B_{gp} = D$. Suppose market clearing gives $\delta p = K_p g + P_z \delta z$, where K_p maps redistribution into equilibrium price changes and P_z gives the direct price effect of the exogenous disturbance. Then $C_v = 0$, $C_g = -K_p$, and $C_p = I$. If shocks enter this household block only through prices, the formulas become

$$\mathbf{A} = \begin{pmatrix} -H & -BK_p \\ C & A + DK_p \end{pmatrix}, \quad f = \begin{pmatrix} -BP_z \\ DP_z \end{pmatrix} \delta z. \quad (3.5c)$$

The upper-right block captures how redistribution changes prices and hence household values. The lower-left block converts changed values into saving and redistribution. DK_p adds the direct effect of equilibrium prices on distribution dynamics. In particular, A alone is the household distribution generator at fixed prices, whereas $\mathbf{A}$ governs the coupled equilibrium response. The grid-based Aiyagari system in (4.21) is a concrete realization of this construction.

Retaining initial and boundary conditions. If the only predetermined variable is the distribution, the inherited condition $g(0) = g_0$ becomes $\Gamma = (0 \ I)$ and $d = g_0$. Other predetermined aggregate states add components to u and rows to Γ . The distribution remains in $\mathcal{M}_0$, and the spatial boundary conditions and infinite-horizon admissibility remain part of the domain. Eliminated prices are reconstructed from (3.5a). Under valid elimination, the homogeneous equilibrium paths therefore correspond to

$$\ker \mathcal{D}_{\text{eq}} \simeq \{u : \dot{u} = \mathbf{A}u, \Gamma u(0) = 0, u \text{ admissible}\}. \quad (3.5d)$$

Thus $\mathbf{A}$ supplies the dynamic coefficients of the reduced path problem. The full residual also contains the time derivative and the boundary restrictions.¹ Its kernel is not $\ker \mathbf{A}$.

A stable initial direction generates a decaying homogeneous response. An unstable direction generates an explosive one unless offset by the response to future shocks. An *exponential dichotomy* is a stable–unstable separation with a uniform positive distance from zero growth. Its projections P_s and P_u select those two components, and E_s denotes all admissible homogeneous initial combinations. The word *trace* below simply refers to the initial value of a path. Thus the

¹For a continuous-space HJB–KFE operator, boundedness of $\mathbf{A}$ is usually unavailable. The same proof works if a Green operator $\mathcal{K} : L^2 \rightarrow \mathcal{X}$ with a bounded initial trace exists, and all admissible homogeneous paths are parameterized continuously by E_s . Establishing these maximal-regularity, domain, and dichotomy properties for the chosen model is an additional analytical task to do.

stable-trace map Γ_s asks which predetermined initial values can be generated by nonexplosive equilibrium dynamics.

Assumption 2 (Dichotomy realization). *The operator A is bounded on a Hilbert space E and has complementary bounded invariant projections P_s, P_u , with $P_s + P_u = I$, such that for $t \geq 0$,*

$$\|e^{tA}P_s\| \leq Me^{-\alpha t}, \quad \|e^{-tA}P_u\| \leq Me^{-\alpha t}, \quad \alpha > 0. \quad (3.6)$$

The trace map $\Gamma_s = \Gamma|_{E_s} : E_s \rightarrow D$, where $E_s = \text{Ran } P_s$, has closed range. When finite defect dimensions are claimed, Γ_s is assumed Fredholm.

For $f \in L^2(\mathbb{R}_+; E)$, define

$$\begin{aligned} (\mathcal{K}f)(t) &= \int_0^t e^{A(t-r)}P_s f(r)dr - \int_t^\infty e^{A(t-r)}P_u f(r)dr, \\ r_f &= (\mathcal{K}f)(0) = - \int_0^\infty e^{-Ar}P_u f(r)dr. \end{aligned} \quad (3.7)$$

The negative-time exponential is used only on the unstable subspace, where it decays. These integrals are bounded maps into the relevant solution and trace spaces. The first integral accumulates past forcing in stable directions. The second works backward from admissibility at infinity to determine today's unstable component from future forcing. Consequently, r_f is the initial adjustment forced by the shock, and the remaining freedom is a stable homogeneous response. This is the forward-looking economic content of the Green operator $\mathcal{K}$.

Theorem 2 (Stable-trace closure criterion). *Under Assumption 2, all $H^1 \cap L^2$ solutions of (3.5) are*

$$u(t) = e^{tA}s + (\mathcal{K}f)(t), \quad s \in E_s, \quad \Gamma_s s = d - \Gamma r_f. \quad (3.8)$$

Thus existence for (f, d) is equivalent to $d - \Gamma r_f \in \text{Ran } \Gamma_s$, and uniqueness is equivalent to $\ker \Gamma_s = \{0\}$. For the full boundary-value operator

$$\mathcal{T}u = (\dot{u} - Au, \Gamma u(0)), \quad (3.9)$$

its kernel and cokernel are isomorphic to those of Γ_s . If Γ_s is Fredholm, so is $\mathcal{T}$, with the same index. In finite dimension, with $n_s = \dim E_s$ and $q = \dim D$,

$$\begin{aligned} \dim \ker \mathcal{T} &= n_s - \text{rank } \Gamma_s, \\ \dim \text{coker } \mathcal{T} &= q - \text{rank } \Gamma_s, \\ \text{ind } \mathcal{T} &= n_s - q. \end{aligned} \quad (3.10)$$

Proof. Exponential decay and Young's inequality show that $\mathcal{K}f \in L^2$. Since A is bounded, its derivative $A\mathcal{K}f + f$ is in L^2 . The difference between any solution and $\mathcal{K}f$ is homogeneous. Its unstable initial component must vanish for the path to belong to L^2 , so that difference is $e^{tA}s$ with $s \in E_s$. The boundary equation gives (3.8). The bounded correspondence $u \leftrightarrow (f, s)$ identifies the domain with $L^2 \times E_s$. Applying the bounded output transformation $(f, d) \mapsto (f, d - \Gamma r_f)$ reduces $\mathcal{T}$ to $I \oplus \Gamma_s$. Kernel, cokernel, closed range, and index follow. $\square$

The theorem gives a geometric interpretation: the affine admissible-trace set $r_f + E_s$ must meet the affine initial-data set $\{u_0 : \Gamma u_0 = d\}$. A single intersection gives a unique solution, an empty intersection gives nonexistence, and a positive-dimensional intersection gives linear indeterminacy. In finite dimension, having $n_s = q$ is only a dimension count. The map Γ_s must also have full rank.

The condition $d - \Gamma r_f \in \text{Ran } \Gamma_s$ asks whether the economy can reconcile its inherited state with the announced future shocks. The reference response $\mathcal{K}f$ already satisfies the dynamic equations and the infinite-horizon condition, but its initial predetermined component Γr_f need not equal the actual inherited data d . The remaining admissible adjustment $e^{tA}s$ must supply precisely the difference $d - \Gamma r_f$. The range of Γ_s lists the initial-state corrections that such adjustments can supply. Thus existence means that the required correction is attainable.

Example 1 (Income, valuation, and the geometry of determinacy). *Let $x(t)$ be the deviation of an income-generating state from its steady state, and let $y(t)$ be the deviation of the value of a claim paying $\kappa x(t)$. The income state decays at rate $\alpha > 0$, and the required return is $\rho > 0$, with $\kappa > 0$. Income capacity is inherited. The asset valuation can jump. The pricing equation equates the required return to the payout plus the capital gain. Require admissible deviations to belong to $H^1 \cap L^2$, as in Theorem 2.*

A stable line in income-valuation space. *Retaining both variables gives*

$$\begin{aligned} \dot{x} &= -\alpha x, & \dot{y} &= \rho y - \kappa x, & x(0) &= x_0, \\ u &= \begin{pmatrix} x \\ y \end{pmatrix}, & A &= \begin{pmatrix} -\alpha & 0 \\ -\kappa & \rho \end{pmatrix}, & f &= 0, & \Gamma &= (1 \ 0), & d &= x_0. \end{aligned} \quad (3.10a)$$

Write $Q = \kappa/(\rho + \alpha)$. The eigenvalues are $-\alpha$ and ρ , and the invariant subspaces and their projections are

$$\begin{aligned} E_s &= \text{span} \left\{ \begin{pmatrix} 1 \\ Q \end{pmatrix} \right\}, & E_u &= \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \\ P_s \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ Qx \end{pmatrix}, & P_u \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ y - Qx \end{pmatrix}. \end{aligned} \quad (3.10b)$$

The stable line $y = Qx$ consists of income and valuation combinations that can jointly return to steady state. Its slope is the present value of the payouts generated by one unit of income capacity: discounting and income decay enter together through $\rho + \alpha$. Stability is therefore a property of a combination of variables. The unstable component is the valuation discrepancy $y - Qx$, not the valuation y alone.

For an arbitrary initial valuation y_0 , the solution is

$$\begin{aligned} x(t) &= x_0 e^{-\alpha t}, \\ y(t) &= Qx_0 e^{-\alpha t} + (y_0 - Qx_0) e^{\rho t}. \end{aligned} \quad (3.10c)$$

Admissibility eliminates the growing valuation discrepancy and selects $y_0 = Qx_0$. Geometrically, the inherited-data set $\{(x, y) : x = x_0\}$ is a vertical line, whereas E_s is the sloping line $y = Qx$. Their unique intersection (x_0, Qx_0) is the initial configuration from which the inherited

income capacity and the asset's fundamental value can evolve consistently. Economically, the intersection selects the impact valuation backed by the entire future payout stream. It is generally away from the steady state. This is the first panel of Figure 1: the dashed orange line fixes income capacity, the blue line imposes the present-value relation, and the black point selects the unique admissible response.

Since $f = 0$, the Green response and its initial trace are zero. The stable-trace map sends $(a, Qa) \in E_s$ to its inherited component a . It is bijective, so $\Gamma_s s = x_0$ uniquely selects $s = (x_0, Qx_0)$ for every inherited income state.

Reduced system: income becomes forcing. Alternatively, solve $x(t) = x_0 e^{-\alpha t}$ first and treat that path as given in the scalar valuation equation. Now $A = \rho$, $E_s = \{0\}$, and $P_u = I$, but the forcing is nonzero:

$$\begin{aligned} \dot{y}(t) &= \rho y(t) + f(t), & f(t) &= -\kappa x_0 e^{-\alpha t}, \\ (\mathcal{K}f)(t) &= h_u(t) = - \int_t^\infty e^{\rho(t-r)} f(r) dr = \int_t^\infty e^{-\rho(r-t)} \kappa x(r) dr = Qx(t), \\ r_f &= Qx_0, & r_f + E_s &= \{Qx_0\}. \end{aligned} \quad (3.10d)$$

The Green operator computes the fundamental value of the remaining payouts. Here $h_u(t)$ denotes the forced unstable component, as in (4.30). The extra term $(y_0 - Qx_0)e^{\rho t}$ in (3.10c) is the freely chosen explosive discrepancy. In the full system, $h_u(t) = 0$ because income is already part of the dynamic state and $f = 0$. The same fundamental valuation appears in the stable line instead.

Removing x moves its inherited condition into the forcing. It does not create an inherited restriction on y . Thus the reduced problem has initial-data space $D = \{0\}$, and $\Gamma_s : \{0\} \rightarrow \{0\}$ is bijective. In the one-dimensional valuation space, the initially permitted set is all of $\mathbb{R}$, while the admissible set is the single point $\{Qx_0\}$. Their intersection again selects a unique valuation. A zero stable subspace means that no admissible homogeneous adjustment can be added to the forced response. As x_0 varies, the pairs (x_0, Qx_0) recover the blue line in the first panel of Figure 1. The reduced problem with a fixed x_0 selects one point on that line.

If an additional restriction also prescribes $y(0) = y_0$, the reduced initial-data map becomes $\Gamma = I$ and $d = y_0$. Existence then requires

$$y_0 = r_f = \int_0^\infty e^{-\rho t} \kappa x(t) dt = Qx_0. \quad (3.10e)$$

No stable homogeneous direction can repair a failure of this condition. With a freely adjusting valuation, the equation determines y_0 .

Eliminating income before valuation is legitimate here because income evolves independently of the asset value. In a heterogeneous-agent equilibrium, distribution dynamics and forward-looking choices generally interact, so the relevant stable space must retain that equilibrium feedback.

Existence and uniqueness ask different economic questions. Existence asks whether any admissible adjustment can correct the required initial mismatch. Uniqueness asks whether

two different admissible adjustments can correct the same mismatch. For a particular experiment, only its own $d - \Gamma r_f$ must belong to $\text{Ran } \Gamma_s$. Surjectivity of Γ_s is stronger: every inherited-state mismatch can then be accommodated. Example 1 permits every inherited income state when the valuation can adjust. Prescribing that valuation as well removes the adjustment margin and creates the compatibility restriction (3.10e). In a larger model, the existence test asks whether the available adjustments make the inherited economy and the announced future mutually consistent.

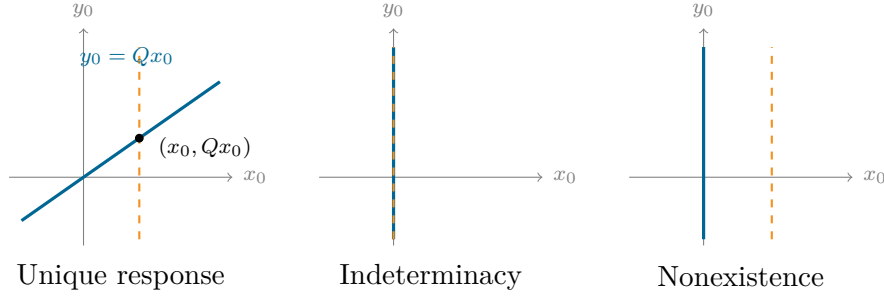


Figure 1: Equilibrium as saddle-path matching. Blue lines show admissible homogeneous initial combinations; dashed orange lines impose the inherited state. The first panel represents Example 1: the stable line $y_0 = Qx_0$ pairs income capacity with its fundamental valuation, and the black intersection selects the unique response. The other panels depict a different system whose stable line is vertical. Coincident lines leave a free response; parallel distinct lines give nonexistence. The last two panels share the same stable dynamics and differ only in the prescribed initial data.

3.3 Adjoint compatibility and the economic source of nonexistence

The existence condition in Theorem 2 generalizes the present-value restriction in Example 1 when the initial valuation is also prescribed. Discounting the pricing equation, integrating over time, and using admissibility equates that valuation with future payouts. The same accounting logic produces an intertemporal government budget constraint by equating inherited debt with future primary surpluses. In a coupled equilibrium system, $\mathbf{A}$ describes several interacting accumulation and adjustment mechanisms. A vector of time-dependent weights takes the place of the scalar discount factor. The purpose remains the same: eliminate the endogenous path from the flow equations and obtain a restriction on inherited data and the prescribed future forcing.

Work in a finite-dimensional realization under Assumption 2, and let $\ell \in \ker \Gamma_s^\top$. The vector ℓ assigns weights to the predetermined data d . Its defining property is $\ell^\top \Gamma_s s = 0$ for every $s \in E_s$: with the forcing held fixed, no admissible homogeneous adjustment can change this weighted combination of initial data. Thus ℓ identifies an inherited balance that the available endogenous adjustments cannot repair. In the constrained variant of Example 1, it is an externally fixed valuation when the entire income path is already prescribed.

Adjoint weights as generalized discounting. Choose weights on the dynamic equations that start at $\psi_\ell(0) = \Gamma^\top \ell$ and evolve according to $\dot{\psi}_\ell = -\mathbf{A}^\top \psi_\ell$. Their path is

$$\psi_\ell(t) = e^{-\mathbf{A}^\top t} \Gamma^\top \ell. \quad (3.11)$$

The initial choice ensures that the weighted initial position is $\ell^\top d$, independently of unknown initial jump variables. The adjoint evolution cancels the internal dynamics when differentiating the weighted position $\psi_\ell(t)^\top u(t)$, just as discounting cancels the required-return term in the pricing equation. Because $\Gamma^\top \ell$ annihilates the stable subspace, these weights decay, even though $-\mathbf{A}^\top$ need not be stable on the entire space. Each component of $\psi_\ell(t)$ measures how forcing in that equation at date t enters the initial compatibility balance. These accounting weights are implied by the model. In a general equilibrium system, they need not be positive or equal to an interest-rate discount factor or a welfare valuation.

The range condition in (3.8) is equivalent to

$$\ell^\top d + \int_0^\infty \psi_\ell(t)^\top f(t) dt = 0 \quad \text{for every } \ell \in \ker \Gamma_s^\top. \quad (3.12)$$

The first term is the weighted inherited position; the second is the corresponding weighted contribution of the entire forcing path. Neither contains the unknown equilibrium path. If one of these balances fails, no choice of initial jump variables and subsequent admissible adjustment can reconcile the data. Depending on the model, the restriction can express fiscal backing, a missing policy adjustment, or inconsistent aggregate initial data.

From flow equations to an intertemporal restriction. Integration of the weighted dynamics, with the boundary term at infinity equal to zero, gives

$$\int_0^\infty \psi_\ell(t)^\top (\dot{u}(t) - \mathbf{A}u(t)) dt = -\psi_\ell(0)^\top u(0) = -\ell^\top d. \quad (3.13)$$

The minus sign is the initial position subtracted from the zero terminal position. Substituting $\dot{u} - \mathbf{A}u = f$ yields (3.12). For the reduced income-valuation problem with $y(0) = y_0$ additionally prescribed, $\mathbf{A} = \rho$, $\Gamma = I$, $d = y_0$, and $f = -\kappa x$. Taking $\ell = 1$ gives $\psi_\ell(t) = e^{-\rho t}$, and (3.12) becomes $y_0 - \int_0^\infty e^{-\rho t} \kappa x(t) dt = 0$, exactly (3.10e). If the valuation can jump freely, this is instead the equation selecting its initial value, and there is no compatibility restriction on x_0 . Appendix C.4 supplies the detailed derivation and proves equivalence to the stable-trace range condition.

These are existence restrictions already contained in the equilibrium problem, not additional assumptions. If Γ_s is onto, its left kernel is zero and there are no independent compatibility restrictions on (f, d) . Uniqueness is a separate question, governed by the right kernel $\ker \Gamma_s$. The adjoint null vector of the full boundary-value operator is the pair (ψ_ℓ, ℓ) : it weights both the dynamic equations and the inherited-data equation. Omitting the latter loses the initial position, just as a discounted flow budget without inherited debt would miss the intertemporal budget constraint.

3.4 Frequency symbols, indices, and exceptional failures

For constant coefficients, an exponential ansatz $u(t) = e^{st}u_s$ gives the pencil $sI - A$. The spatial generator can have infinitely many modes before discretization. After a finite spatial discretization, transfer functions such as $K(sI - A)^{-1}D$ are rational wherever the inverse exists.

Indeed, the unilateral Laplace transform gives

$$(sI - A)\hat{u}(s) = u(0) + \hat{f}(s), \quad \hat{u}(s) = \int_0^\infty e^{-st}u(t)dt. \quad (3.14)$$

The transform is initially defined for $\text{Re } s > 0$ for the admissible paths considered here. Forward-looking components of $u(0)$ are unknown and must be chosen to eliminate inadmissible poles. Pointwise invertibility of the pencil on the imaginary axis can establish hyperbolicity in finite dimension. In infinite dimension, even spectral separation generally requires additional resolvent or semigroup bounds to yield a dichotomy.

Example 2 (Zero index with both defects). *Take $A = \text{diag}(-1, 1)$ and $\Gamma = \begin{pmatrix} 0 & 1 \end{pmatrix}$. The imaginary-axis pencil is invertible, and $n_s = q = 1$. Nevertheless $\Gamma_s = 0$. With $f = 0$, a nonzero prescribed second component has no admissible solution; with zero prescribed second component, the first component is arbitrary and decays. Kernel and cokernel dimensions both equal one. Changing the boundary map to $\Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}$ makes the same dynamics determinate.*

This example also explains the role of genericity in index criteria. The winding-number formulas in Auclert et al. (2025) identify an index and, under their additional scalar or genericity conditions, determine the individual defects. An index alone determines their difference. Our stable-trace test retains the boundary orientation directly. It is therefore compatible with quasi-Toeplitz index theory. A continuous-time winding formula would additionally require precise symbol classes, asymptotics, and a choice of contour orientation.

3.5 Sequence-space reduction and local nonlinear equilibrium

The stable-space calculation is an internal realization of the sequence-space equilibrium. Household value and distribution paths can be eliminated to retain only aggregate paths as equilibrium unknowns. Analytical fake news computes the derivatives of that elimination, as developed in Appendix D.1. Write w_{int} for internal *deviation* paths and retain δp as aggregate unknowns:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{E} \end{pmatrix} \begin{pmatrix} w_{\text{int}} \\ \delta p \end{pmatrix} = \begin{pmatrix} b_w \\ b_p \end{pmatrix}. \quad (3.15)$$

Here $\mathcal{A}$ is the internal *boundary-value* operator, not only its interior generator. If $\mathcal{A}$ is boundedly invertible, eliminating w_{int} gives the aggregate SSJ closure

$$\mathcal{J}_{\text{GE}}\delta p = b_p - \mathcal{C}\mathcal{A}^{-1}b_w, \quad \mathcal{J}_{\text{GE}} = \mathcal{E} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}. \quad (3.16)$$

Proposition 1 (Equivalence of full and reduced closure). *If $\mathcal{A}$ is boundedly invertible, the full operator in (3.15) is equivalent by bounded invertible row and column operations to $\text{diag}(\mathcal{A}, \mathcal{J}_{\text{GE}})$. Therefore kernel, cokernel, and Fredholm index of the equilibrium problem equal those of the aggregate SSJ closure, up to canonical isomorphisms.*

Proof. Left multiplication by $\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}$, followed by right multiplication by $\begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}$, gives the claimed diagonal operator. $\square$

Selecting a convenient set of prices changes the representation and possibly the quality of a preconditioner. It preserves determinacy only when the resulting elimination and reconstruction maps are valid.

For nonlinear equilibrium, suppose $\mathcal{F} : \mathcal{X} \times \mathcal{Z} \rightarrow \mathcal{Y}$ is continuously Frechet differentiable on an open chart near the reference equilibrium. If $D_x \mathcal{F}$ is boundedly invertible, the implicit-function theorem gives a locally unique differentiable equilibrium map $x(z)$. Its derivative solves (2.11). This local statement is stronger than a formal linearization but weaker than global uniqueness. A nontrivial linear kernel signals a failure of this sufficient regularity condition. At nonsmooth constraints or probability-boundary points, an appropriate constrained or generalized implicit-function theorem would be needed.

4 Equilibrium determinacy and computation through stable subspaces

Section 3 characterizes existence and uniqueness through the stable-trace map. We now construct the required split from household response factors and use it to compute equilibrium responses. The same projection supplies the determinacy test, the calibrated Aiyagari IRFs, and the continuation condition for finite-prefix SSJ calculations. Sections 4.1–4.2 develop the general method. Sections 4.3–4.5 carry out one seven-step Aiyagari calculation, followed by the SSJ comparison and an extension to nominal feedback.

4.1 Household dynamics and equilibrium price feedback

Theorem 2 turns an infinite-horizon question into a boundary calculation for a stationary finite-state realization. The main computational obstacle is constructing its stable space when household values and distributions have many coordinates. The household factors used in fake news provide a way to apply the spectral projections without storing a full stable basis. We first identify the household split and the price-feedback calculation. Section 4.2 uses these objects to construct the equilibrium projection and test the initial-data restrictions. This use of household response objects is consistent with the recovery of state-space dynamics from sequence-space solutions in Auclert et al. (2021), Appendix C.7. The object constructed here is the stable–unstable projection of the complete equilibrium realization.

The fixed-price household split. Work on a fixed spatial grid, suppress its subscript, and consider (3.5c). The distribution coordinates have zero total mass. Assume that A and H are Hurwitz, meaning that all their eigenvalues have negative real parts. The remaining coupling maps are finite matrices. Setting equilibrium price feedback to zero gives

$$A_0 = \begin{pmatrix} -H & 0 \\ C & A \end{pmatrix}, \quad \Xi = \int_0^\infty e^{At} C e^{Ht} dt, \quad A\Xi + \Xi H = -C. \quad (4.1)$$

Here A_0 contains both household equations and it is distinct from the distribution generator A . At fixed prices, stable directions have zero value deviation, while unstable value deviations carry the associated distribution profile Ξv . Thus

$$\begin{aligned}(v, g) &= (0, g - \Xi v) + (v, \Xi v), \\ P_s^0(v, g) &= (0, g - \Xi v), \quad P_u^0(v, g) = (v, \Xi v).\end{aligned}\tag{4.2}$$

These are complementary spectral projections, generally oblique.

The map Ξ records the distributional legacy associated with a value-function deviation at fixed aggregate prices. A change in the marginal value of wealth changes saving incentives. C converts the policy response into a source in the distribution equation. To interpret the integral in (4.1), fix today's value profile v . At a hypothetical earlier date $-\tau$, the homogeneous value profile that evolves into v is $e^{H\tau}v$. The induced distribution source is $Ce^{H\tau}v$, and $e^{A\tau}$ propagates that source to today. Integrating over $\tau \geq 0$ gives Ξv , a signed distribution profile with zero total mass. This hypothetical history defines the invariant graph.

The subtraction $g - \Xi v$ in (4.2) removes the distribution component associated with the value deviation. Indeed, in the homogeneous fixed-price system, the transformed distribution coordinate $\eta = g - \Xi v$ satisfies

$$\dot{\eta} = A\eta, \quad \dot{v} = -Hv, \quad g = \eta + \Xi v.$$

Thus $(0, g - \Xi v)$ describes redistribution that relaxes under the baseline household dynamics, while $(v, \Xi v)$ is the joint value and distribution movement along the unstable graph. Without forcing, the latter component must vanish for the path to be admissible. Both pieces contain a distribution component: simply assigning g to the stable space and v to the unstable space would ignore the saving response encoded in C . Appendix C.6 verifies the cancellation that separates these dynamics.

Fake news as a price-feedback calculation. Let n_p be the number of price inputs. Replace the generic aggregation map K in (D.2) by the equilibrium-price map K_p , and write τ for the anticipation lag to distinguish it from the complex frequency s :

$$\begin{aligned}F_p(\ell, \tau) &= K_p e^{A\ell} C e^{H\tau} B, \\ \widehat{F}_p(s) &= \int_0^\infty \int_0^\infty e^{-s\ell + s\tau} F_p(\ell, \tau) d\tau d\ell \\ &= -K_p (sI - A)^{-1} C (sI + H)^{-1} B, \\ M(s) &= I - K_p (sI - A)^{-1} D - \widehat{F}_p(s).\end{aligned}\tag{4.3}$$

If the household decay rates are $\alpha_A, \alpha_H > 0$, the integrals converge for $-\alpha_A < \operatorname{Re} s < \alpha_H$, including the imaginary axis. The opposite exponential signs reflect forward distribution propagation and backward anticipation.

To interpret $M(s)$, consider a proposed price mode $\delta p(t) = e^{st}p$. The vector p specifies the relative price movements. The real part of s determines growth or decay, and its imaginary part determines oscillation. The three terms in $M(s)$ compare this proposal with the prices implied by the household response:

1. I retains the proposed price vector itself, so $Ip = p$.
2. $K_p(sI - A)^{-1}D$ gives the *direct saving channel*. The proposed prices change income and saving through D . The household distribution responds through $(sI - A)^{-1}$. And K_p maps that redistribution back into equilibrium prices. This channel holds the value-function input fixed when differentiating saving.
3. $\widehat{F}_p(s)$ gives the *value and policy channel*. Anticipated prices change household values, values change consumption and saving, and the resulting distribution movement changes equilibrium prices. The transformed fake-news kernel collects this indirect response across anticipation and distribution-propagation lags.

Consequently,

$$M(s)p = p - \underbrace{\left[K_p(sI - A)^{-1}D + \widehat{F}_p(s) \right] p}_{\text{prices implied by the household response}} .$$

Thus $M(s)p$ is the price-consistency residual: proposed prices minus the prices reproduced by household decisions and redistribution. In Aiyagari (1994), interest rates and wages change saving and hence aggregate capital. Firms' marginal-product conditions then imply new interest rates and wages. Consistency requires that these implied movements match the proposal in both magnitude and timing.

The inverse of $M(s)$, when it exists, measures equilibrium feedback amplification. For a scalar illustration at a given frequency, suppose a proposed price movement of one generates an implied movement of 0.8 with the same phase. Then $M(s) = 0.2$: a unit external price forcing requires $p = 1 + 0.8p$, giving $p = 5$. With several prices, the matrix also accounts for feedback across prices. A small singular value signals a price direction in which this consistency calculation is sensitive to forcing and numerical error.

When the household resolvents exist, $M(s)p = 0$ for some $p \neq 0$ identifies a price pattern reproduced without external forcing, together with its associated household mode. Its growth or decay is determined by $\text{Re } s$. Under the finite-grid assumptions here, Proposition 2 shows that nonsingularity of $M(i\omega)$ for every real ω excludes equilibrium modes on the imaginary axis and permits a stable-unstable split. The small price solve in (4.5) then uses $M(s)$ to reconstruct the full equilibrium resolvent. Integrating its feedback correction in (4.6) constructs the spectral projections.

4.2 Constructing the stable split and testing determinacy

The household response factors and price-consistency matrix now supply the equilibrium resolvent. Its frequency integral constructs the stable projection, and the initial-data map turns that projection into existence and uniqueness tests.

Reconstructing the household resolvent. The aggregate kernel must be retained together with the household maps that reconstruct values and distributions. For each frequency, compute

the n_p -column response matrices

$$\begin{aligned} Y_v(s) &= -(sI + H)^{-1}B, \\ Y_g(s) &= (sI - A)^{-1}\{D + CY_v(s)\}, \\ Y(s) &= \begin{pmatrix} Y_v(s) \\ Y_g(s) \end{pmatrix}, & M(s) &= I - K_p Y_g(s). \end{aligned} \tag{4.4}$$

For an arbitrary input profile $x = (x_v, x_g)$, first solve the household equations at fixed prices, then impose price consistency:

$$\begin{aligned} v^0 &= (sI + H)^{-1}x_v, & g^0 &= (sI - A)^{-1}(x_g + Cv^0), \\ M(s)p &= K_p g^0, & (sI - A)^{-1}x &= \begin{pmatrix} v^0 \\ g^0 \end{pmatrix} + Y(s)p. \end{aligned} \tag{4.5}$$

The superscript 0 denotes this auxiliary fixed-price frequency response, not an initial condition. Each evaluation uses the spatial household matrices and a small price solve.

Proposition 2 (Stable projections from household responses). *Let the real finite-dimensional household matrices A, H be Hurwitz, and let A be given by (3.5c). Then A has no imaginary-axis eigenvalue if and only if $M(i\omega)$ is nonsingular for every $\omega \in \mathbb{R}$. When this holds, define*

$$\begin{aligned} \mathcal{R}_0(s) &= (sI - A_0)^{-1}, \\ \Delta \mathcal{R}(s) &= Y(s)M(s)^{-1}(0 \ K_p)\mathcal{R}_0(s), \\ P_s &= P_s^0 + \frac{1}{\pi} \int_0^\infty \operatorname{Re}\{\Delta \mathcal{R}(i\omega)\}d\omega, \quad P_u = I - P_s. \end{aligned} \tag{4.6}$$

The integral converges in matrix norm and gives the spectral projection onto the stable invariant subspace along the unstable subspace. No smallness restriction on price feedback is required.

Appendix C.6 proves the proposition and gives bounds for quadrature and for certifying frequency separation.

Why the equilibrium split needs a correction. The baseline projection P_s^0 describes household dynamics with aggregate prices held fixed. It already includes the effect of values on saving and redistribution through C and Ξ . General equilibrium closes the return channel: redistribution changes prices, which feed back into values and saving. This changes which combinations of values and distributions converge. At fixed prices, decaying departures have $v = 0$. In Aiyagari, an inherited increase in aggregate capital lowers interest rates and raises wages. Households' current valuations must reflect the subsequent equilibrium price path. A convergent equilibrium transition therefore generally requires an initial value adjustment. The correction $P_s - P_s^0$ incorporates this price feedback into the admissible directions and vanishes when $K_p = 0$.

Computationally, the decomposition reuses the known household split and the response factors underlying fake news. For an input profile x , first compute the fixed-price response $\mathcal{R}_0(s)x = (v^0, g^0)$. Its distribution component implies the price movement $K_p g^0$. Solving

$M(s)p = K_pg^0$ makes the price response consistent with household feedback, and $Y(s)p = \Delta\mathcal{R}(s)x$ reconstructs the additional value and distribution responses. Integrating this correction as in (4.6) gives the action of the equilibrium stable projection. $P_u x = x - P_s x$ gives the complementary action. Thus household solves and small price solves construct the split without storing a full stable basis. These actions enter the initial-data calculation in (4.8) and the continuation condition in (D.26). The baseline term retains household directions that are invisible to price experiments. This is why an aggregate fake-news table alone does not identify the complete stable space.

With N_ω positive-frequency quadrature nodes, the real approximation to $P_s - P_s^0$ has rank at most $2n_p N_\omega$. Its thin factors can be cached or streamed, with storage proportional to $(n_v + n_g)n_p N_\omega$ when cached, where n_v, n_g are the value and distribution dimensions. This bound concerns the quadrature correction. The exact correction can have full rank, and the cost of applying Ξ remains.

Algorithm 1 Applying the equilibrium split with fake-news household factors

Require: Consistent mass-zero household matrices A, H, B, C, D, K_p , an action of Ξ , and numerical error tolerances.

- 1: Verify household stability. Assess $M(i\omega)$ on the imaginary axis using interval and high-frequency bounds when certification is required. (See Appendix A.3, the weighted adjoints and norm scaling in (A.3).)
- 2: Choose frequency quadrature nodes and weights. Factor or precondition the shifted household matrices and form Y_v, Y_g, M at those nodes.
- 3: **for** each input profile x **do**
- 4: Apply $P_s^0 x$ using (4.2).
- 5: At each node solve (4.5) and accumulate the weighted real feedback correction in (4.6).
- 6: Return the approximate $P_s x$ and $P_u x = x - P_s x$.
- 7: **end for**
- 8: Implement adjoint actions from the same factors. Refine quadrature and solve tolerances and check projection, invariance, and boundary residuals.

Ensure: Actions of the equilibrium spectral projections to the assessed tolerance, conditional on the stated spectral and numerical bounds.

From the split to initial-data compatibility. After eliminating the known population-mass direction and regular static equations, an ordered real Schur decomposition provides a direct benchmark basis Q_s for moderate realizations. Let q be the number of independent predetermined initial values and n_s the number of stable directions. When such a basis is stored, form the finite matrix

$$B_s = \Gamma_h Q_s \in \mathbb{R}^{q \times n_s}. \quad (4.7)$$

Multiplying a stable-mode coefficient vector by B_s tells us the initial distribution and other predetermined states implied by that admissible response. This is why the rank of B_s , not the eigenvalue count alone, answers the equilibrium question.

Alternatively, projection actions permit the equivalent constrained problem

$$P_u s_0 = 0, \quad \Gamma s_0 = d - \Gamma r_f. \quad (4.8)$$

Here s_0 is the stable initial adjustment. The first equation restricts it to the stable space. The second matches the inherited data after accounting for future forcing. For $\Gamma = (0 \ I)$ and no forcing, fix $g = g_0$ and solve $P_u(v, g_0) = 0$ for the jump profile v . This supports iterative constrained solves without a stored stable basis.

For the basis representation, use a full singular value decomposition $B_s = U\Sigma V^\top$, or a rank-revealing factorization. If r is the rank, the last $n_s - r$ columns of V give vectors a_j with $B_s a_j = 0$. Each produces a complete homogeneous equilibrium response

$$u_j(t) = e^{tA_h} Q_s a_j. \quad (4.9)$$

These paths obey the economic equations, decay, and leave the predetermined initial conditions unchanged. Reconstructing any eliminated prices or quantities gives the corresponding full GE kernel vectors. Economically, these are the extra movements that the model does not pin down.

The last $q - r$ columns of U give vectors ℓ_j with $B_s^\top \ell_j = 0$. Their associated adjoint paths are $\psi_j(t) = e^{-A_h^\top t} \Gamma_h^\top \ell_j$. For the proposed experiment, compute the finite list of numbers

$$c_j = \ell_j^\top (d - \Gamma_h r_f) = \ell_j^\top d + \int_0^\infty \psi_j(t)^\top f(t) dt. \quad (4.10)$$

All must vanish for an equilibrium response to exist. The ℓ_j identify which combinations of initial data cannot be generated by admissible equilibrium dynamics. The integral adds the effect of the announced shock. A zero-dimensional cokernel means these restrictions are absent.

4.3 Aiyagari: household responses and market clearing

The one-asset Aiyagari economy makes the entire procedure concrete. Households save in productive capital, face idiosyncratic labor-income risk, and cannot borrow below $\underline{a}$. A competitive firm rents capital and hires labor. We study the first-order response to an aggregate productivity path $\delta Z(t)$, announced at date zero, starting from a given distribution. Sections 4.3–4.5 implement all seven steps at one calibrated steady state. This subsection develops household responses and market clearing. The next two realize the closure on a DEC grid and use its stable projection to test determinacy and compute IRFs. Household fake-news factors construct the equilibrium spectral projection, which supplies both the determinacy test and the continuation condition for an aggregate SSJ solve. A full-equilibrium Schur decomposition is used afterwards as an independent numerical benchmark. Appendix C.7 gives the computational derivations, accuracy checks, and reproduction instructions.

Step 1: compute the stationary allocation. Let z_i denote idiosyncratic productivity and let q_{ij} be the entries of its Markov generator, including diagonal entries with zero row sums. Period utility is $U(c)$, with $U' > 0$ and $U'' < 0$. The level HJB is

$$\rho V_i - \partial_t V_i = \max_c \left\{ U(c) + (wz_i + ra - c) \partial_a V_i + \sum_j q_{ij} V_j \right\}. \quad (4.11)$$

Given trial steady-state prices, solve for $\bar{V}_i$, consumption $\bar{c}_i$, and saving $\bar{s}_i = \bar{w}z_i + \bar{r}a - \bar{c}_i$. Their generator $\bar{L}$ determines the stationary distribution through $\bar{L}^*\bar{\mu} = 0$, with total mass one.

Let the capital share be $\theta \in (0, 1)$, depreciation be δ_{dep} , and aggregate labor be fixed at $\bar{N}$. In levels,

$$\begin{aligned} Y &= ZK^\theta \bar{N}^{1-\theta}, \\ r &= \theta ZK^{\theta-1} \bar{N}^{1-\theta} - \delta_{\text{dep}}, \quad w = (1 - \theta)ZK^\theta \bar{N}^{-\theta}. \end{aligned} \quad (4.12)$$

Writing depreciation with a subscript distinguishes that parameter from the perturbation operator δ . Stationary capital is $\bar{K} = \sum_i \int a \bar{\mu}_i(da)$. A finite asset moment is part of the admissible distribution space.

For the numerical illustration, set $U(c) = -1/c$, $\rho = \delta_{\text{dep}} = 0.05$, $\theta = 1/3$, and $\bar{Z} = 0.1$. Income takes values $z_1 = 1, z_2 = 2$, with switching intensities $q_{12} = q_{21} = 1/3$. The stationary income shares are one half each, so $\bar{N} = 3/2$.

Step 2: differentiate household behavior at given prices. At first order, the envelope condition removes the direct effect of δc from the HJB: its utility and saving effects offset at an interior optimum. The derivative of the first-order condition then recovers the consumption response. On a differentiable policy region,

$$\begin{aligned} (\rho - \bar{L} - \partial_t)v_i &= \partial_a \bar{V}_i(a \delta r + z_i \delta w), \\ \delta c_i &= \frac{\partial_a v_i}{U''(\bar{c}_i)}, \quad \delta s_i = a \delta r + z_i \delta w - \delta c_i, \\ \dot{g}_i &= (\bar{L}^* g)_i - \partial_a(\bar{\mu}_i \delta s_i). \end{aligned} \quad (4.13)$$

The input $\delta p = (\delta r, \delta w)^\top$ enters the value equation through known payoff exposures $a \partial_a \bar{V}_i$ and $z_i \partial_a \bar{V}_i$. A higher expected wage or interest rate can therefore change consumption and saving before that price change occurs. The KFE translates the saving response into a redistribution of households. The response g has positive and negative components and integrates to zero. The state-constraint and flux conditions are differentiated consistently with these interior equations. At a borrowing boundary or switching point, we use the appropriate one-sided/weak response. The displayed division by U'' describes the smooth policy region.

To make the link with analytical fake news explicit, let B_r and B_w be the two payoff-exposure columns and let $H = \bar{L} - \rho I$. The consumption response to a price change $j \in \{r, w\}$ occurring τ units ahead uses the policy derivatives C_c, D_c defined in Section 2:

$$\begin{aligned} \phi_{c,j}(\tau) &= C_c e^{H\tau} B_j, \\ \delta c(t) &= D_c \delta p(t) + \sum_{j \in \{r, w\}} \int_0^\infty \phi_{c,j}(\tau) \delta p_j(t + \tau) d\tau. \end{aligned} \quad (4.14)$$

On a smooth interior policy region, $C_c v = \partial_a v / U''(\bar{c})$. The calibrated calculation keeps the upwind active set fixed when differentiating. At zero-drift nodes the corresponding rule is $\delta c = a \delta r + z_i \delta w$, so the direct term $D_c \delta p$ is retained. This is a contemporaneous constrained policy response, distinct from anticipated news.

For anticipated saving news the induced distributional source is $\mathcal{D}_{\tau,j} = \partial_a(\bar{\mu} \phi_{c,j}(\tau))$, because the induced saving change is $-\delta c$. The direct source is $-\partial_a[\bar{\mu}(a \delta r + z_i \delta w - D_c \delta p)]$. These are

the C and D channels in (2.8), with their discrete counterparts obtained from the same saving flux. Independent perturbations of values and prices check both the HJB and KFE derivatives, including the active boundary rule.

Step 3: construct asset-supply Jacobians. Choose the observable $\varphi(a, i) = a$. Propagate it with steady-state household transitions and pair it with each source $\mathcal{D}_{s,j}$, as in (D.3). Integrating those fake-news contributions over the adjustment date and adding the direct source gives $\mathcal{J}_{Kr}$ and $\mathcal{J}_{Kw}$. Thus

$$\delta K^s = \mathcal{J}_{Kr}\delta r + \mathcal{J}_{Kw}\delta w + k_{\text{ini}}, \quad k_{\text{ini}}(t) = \left\langle a, e^{t\bar{L}^*} g(0) \right\rangle. \quad (4.15)$$

The superscript s means household asset supply. Capital here is an aggregate variable, whereas the K of the generic block was the aggregation map. For an unanticipated productivity experiment starting at the stationary distribution, $g(0) = 0$, so $k_{\text{ini}} = 0$. Prices and value functions can jump at the announcement date, but the inherited household asset holdings cannot.

Using the grid coefficients defined above, the two dated kernels are particularly explicit:

$$\begin{aligned} F_j(\ell, \tau) &= k_h^\top e^{A_{0,h}\ell} C_{0,h} e^{H_h\tau} B_{h,j}, \\ J_{Kj,h}(t, s) &= \mathbf{1}_{\{s < t\}} k_h^\top e^{A_{0,h}(t-s)} D_{0,h,j} + \int_0^{\min(t,s)} F_j(t-u, s-u) du, \quad j \in \{r, w\}. \end{aligned} \quad (4.16)$$

Anticipation changes saving at an adjustment date u and the forward generator then carries the resulting redistribution to the observation date t . Integrating over admissible adjustment dates gives the household asset-supply Jacobian.

Step 4: close asset supply with firm pricing. Differentiate (4.12) to obtain

$$\delta r = r_K \delta K + r_Z \delta Z, \quad \delta w = w_K \delta K + w_Z \delta Z, \quad (4.17)$$

where

$$\begin{aligned} r_K &= (\theta - 1) \frac{\bar{r} + \delta_{\text{dep}}}{\bar{K}}, & r_Z &= \frac{\bar{r} + \delta_{\text{dep}}}{\bar{Z}}, \\ w_K &= \theta \frac{\bar{w}}{\bar{K}}, & w_Z &= \frac{\bar{w}}{\bar{Z}}. \end{aligned} \quad (4.18)$$

More capital lowers its marginal product and raises the wage. Those price changes feed back into household saving. Setting $\delta K^s = \delta K$ gives the aggregate sequence-space equilibrium equation

$$\begin{aligned} \mathcal{J}_{\text{GE}}^K \delta K &= b_K, \\ \mathcal{J}_{\text{GE}}^K &= I - r_K \mathcal{J}_{Kr} - w_K \mathcal{J}_{Kw}, \\ b_K &= (r_Z \mathcal{J}_{Kr} + w_Z \mathcal{J}_{Kw}) \delta Z + k_{\text{ini}}. \end{aligned} \quad (4.19)$$

This is an explicit closure operator. Its kernel contains capital paths reproduced by feedback between households and firms even with no shock and no initial redistribution. Its cokernel identifies productivity or initial-distribution effects that the available endogenous capital adjustment cannot accommodate. Solving (4.19) determines δK . Factor prices follow from (4.17), and household policies and the distribution follow from (4.13). Consumption and output are then computed by aggregation and production.

4.4 Aiyagari: DEC realization and stable projection

The preceding closure is expressed through continuous-state household operators. Steps 5 and 6 construct its conservative spatial realization and recover the equilibrium split through scalar capital feedback.

Step 5: realize the same closure on a DEC grid. Let $\bar{L}_h$ be the steady-state backward generator on n individual-state nodes. Use a matrix Z_m with orthonormal columns spanning the admissible distribution subspace and write $\delta m_h = Z_m \xi$. Imposing total mass alone gives $q = n - 1$ independent coordinates. Here we also hold both income-state shares at their stationary values, consistently with fixed aggregate labor: perturbations sum to zero within each income state. This invariant subspace has $q = n - 2 = 2(N_a - 1)$ coordinates. Allowing income-share perturbations would require including their aggregate labor response in firm pricing. Denote by $C_h v$ the policy-induced cell-mass source and by $D_h \delta p$ the direct price-induced source, constructed by differentiating the same generator family. Define

$$\begin{aligned} H_h &= \bar{L}_h - \rho I, & A_{0,h} &= Z_m^\top \bar{L}_h^\top Z_m, \\ C_{0,h} &= Z_m^\top C_h, & D_{0,h} &= Z_m^\top D_h, \\ k_h^\top &= a_h^\top Z_m, & P_K &= (r_K, w_K)^\top, & P_Z &= (r_Z, w_Z)^\top. \end{aligned} \quad (4.20)$$

The two columns of B_h are the discrete HJB payoff derivatives. Their formula and the policy derivative must come from the same discretized optimality conditions. Simply combining unrelated derivative stencils would not ensure consistency. Then $\delta K = k_h^\top \xi$ and $\delta p = P_K k_h^\top \xi + P_Z \delta Z$. Substitution gives the complete reduced dynamic system

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} v_h \\ \xi \end{pmatrix} &= \underbrace{\begin{pmatrix} -H_h & -B_h P_K k_h^\top \\ C_{0,h} & A_{0,h} + D_{0,h} P_K k_h^\top \end{pmatrix}}_{A_{GE,h}} \begin{pmatrix} v_h \\ \xi \end{pmatrix} + \underbrace{\begin{pmatrix} -B_h P_Z \\ D_{0,h} P_Z \end{pmatrix}}_{b_{Z,h}} \delta Z, \\ \Gamma_h \begin{pmatrix} v_h(0) \\ \xi(0) \end{pmatrix} &= \xi_0, \quad \Gamma_h = (0 \quad I), \quad \xi_0 = Z_m^\top \delta m_h(0). \end{aligned} \quad (4.21)$$

The signs in the upper row follow from $-\dot{v}_h = H_h v_h + B_h \delta p$. The lower row is the conservative KFE. This matrix and (4.19) represent the same household–firm closure whenever household elimination is valid. The matrix representation is convenient for explaining and testing stable directions. The aggregate SSJ representation can be preferable for large-scale solution.

For $N_a = 100$, there are 200 value coordinates, 198 distribution coordinates, and a 398-dimensional equilibrium realization. The numerical kernels in Steps 2–3 use these same operators. The implementation checks constant preservation, source-mass conservation, and invariance of the restricted distribution space. The full matrix is available for residual checks and moderate-grid time propagation, while the split below is constructed from household feedback.

Step 6: construct the split through capital feedback. The general price calculation in Section 4.1 simplifies further here. Although households respond to two prices, their endogenous

changes both come from one scalar, aggregate capital: $K_p = P_K k_h^\top$. Consequently,

$$\mathbf{A}_{\text{GE},h} = \begin{pmatrix} -H_h & 0 \\ C_{0,h} & A_{0,h} \end{pmatrix} + \begin{pmatrix} -B_h P_K \\ D_{0,h} P_K \end{pmatrix} (0 \quad k_h^\top). \quad (4.22)$$

The price-feedback perturbation has rank at most one, although its effect on the stable space need not have rank one. The relevant fake-news kernel measures the asset-supply response to the price changes induced by a unit of trial capital:

$$F_K(\ell, \tau) = k_h^\top e^{A_{0,h}\ell} C_{0,h} e^{H_h \tau} B_h P_K. \quad (4.23)$$

Use the signed transform in (4.3) for $\widehat{F}_K(s)$. Since $F_K = F_r r_K + F_w w_K$, its transform can be evaluated directly by household shifted solves:

$$\widehat{F}_K(s) = -k_h^\top (sI - A_{0,h})^{-1} C_{0,h} (sI + H_h)^{-1} B_h P_K. \quad (4.24)$$

This evaluates the transform of the same fake-news kernel used in Step 3 without numerical integration over a truncated two-dimensional time window. The household shifted solves give

$$\begin{aligned} Y_{v,K}(s) &= -(sI + H_h)^{-1} B_h P_K, \\ Y_{g,K}(s) &= (sI - A_{0,h})^{-1} \{D_{0,h} P_K + C_{0,h} Y_{v,K}(s)\}, \\ m_K(s) &= 1 - k_h^\top Y_{g,K}(s) \\ &= 1 - k_h^\top (sI - A_{0,h})^{-1} D_{0,h} P_K - \widehat{F}_K(s). \end{aligned} \quad (4.25)$$

Thus the price-consistency step is a scalar division. Economically, $m_K(s)$ subtracts the capital supplied by households from the capital movement proposed to firms, including both anticipated policy changes and the direct saving response. It contains the stationary feedback underlying (4.19). The complete half-line SSJ also retains its inherited distribution and initial-boundary correction.

For an input $x = (x_v, x_\xi)$, let (v^0, ξ^0) be the fixed-price resolvent response in (4.5), using $H_h, A_{0,h}, C_{0,h}$. The full feedback correction is

$$\Delta \mathcal{R}_h(s)x = \begin{pmatrix} Y_{v,K}(s) \\ Y_{g,K}(s) \end{pmatrix} \frac{k_h^\top \xi^0}{m_K(s)}. \quad (4.26)$$

Construct the baseline projection with $A_{0,h}\Xi_h + \Xi_h H_h = -C_{0,h}$, and integrate this correction as in (4.6). This gives actions of $P_{s,h}, P_{u,h}$ without a full eigendecomposition of $\mathbf{A}_{\text{GE},h}$. At N_ω positive-frequency nodes, the real feedback correction has rank at most $2N_\omega$. The household solves, baseline action, and conditioning still determine its actual cost. A zero of $m_K(i\omega)$ prevents this split, while a small value calls for finer accuracy and assessment of proximity to a spectral crossing.

Mapped Gauss-Legendre quadrature with 32, 64, and 128 positive-frequency nodes constructs the projection numerically. At the finest quadrature on the $N_a = 100$ grid, its relative idempotence residual is 1.20×10^{-9} . An SVD of the computed projection supplies an orthonormal basis Q_s for its range. There are 198 retained singular values, all at least approximately one, and the largest discarded singular value is below 9×10^{-11} . The change in the orthogonal

range projector between 64 and 128 nodes is 3.94×10^{-8} . These are numerical convergence diagnostics. The present moderate-grid calculation materializes the projection. Larger implementations can use its actions in the projected boundary equations without storing a full stable basis.

4.5 Aiyagari: determinacy and impulse responses

We use the same spatial realization and computed projection to test the initial-data restrictions and recover the productivity response.

Step 7: test determinacy and compute the response. The $q = n - 2$ distribution coordinates of the calibrated fixed-labor economy are predetermined, while the value response is free to adjust. Form $B_s = \Gamma_h Q_s$ using the range obtained in Step 6. Equivalently, impose $P_{u,h}s_0 = 0$ and $\Gamma_h s_0 = \xi_0 - \Gamma_h r_f$ through projection actions. Table 2 reports the boundary test. The number of stable directions equals the number of predetermined coordinates, and B_s has full rank on both grids. Thus both the kernel and cokernel defects vanish numerically: each admissible initial distribution and forcing has one compatible stable adjustment in this finite-grid model. In another calibration, the procedure must retain any null directions and compatibility restrictions it finds.

Table 2: Calibrated Aiyagari determinacy from the fake-news projection

N_a	n	q	n_s	rank B_s	$\sigma_{\min}(B_s)$
50	100	98	98	98	1.69674×10^{-5}
100	200	198	198	198	8.72229×10^{-6}

Notes: N_a counts asset nodes per income state. Both income shares are fixed, giving $q = n - 2$. The basis is extracted from the computed spectral projection. Singular values use Euclidean norms in value and cell-probability coordinates. Their levels are sensitive to units and do not establish a uniform continuous-state conditioning bound.

For the productivity experiment, set $\delta Z(t) = Z_0 e^{-\nu t}$, with $Z_0 = 0.001$, $\nu = 0.2$, and $\xi_0 = 0$. This is a one-percent initial TFP increase. The same household-feedback resolvent evaluated at $s = -\nu$ gives

$$r_f = P_{u,h}(-\nu I - A_{GE,h})^{-1} b_{Z,h} Z_0. \quad (4.27)$$

The displayed full inverse is evaluated through the household shifted solves and scalar division in (4.25)–(4.26). It exists at the stated calibration and decay rate. The unstable-subspace formula in (D.18) remains available when a stable forcing resonance prevents this particular full-resolvent evaluation. Solving for the stable initial trace then determines the entire response path, as follows.

Write $u_h(t) = (v_h(t), \xi(t))$ and $f_h(t) = b_{Z,h} \delta Z(t)$. Let $T_s = Q_s^\top A_{GE,h} Q_s$ be the equilibrium generator restricted to the orthonormal stable basis, so that $A_{GE,h} Q_s = Q_s T_s$. The vector $c \in \mathbb{R}^{n_s}$ contains the constant coefficients of the stable initial adjustment: if q_j is column

j of Q_s , then $Q_s c = \sum_j c_j q_j$. Here c denotes coefficients, whereas $c_i(t, a)$ elsewhere denotes household consumption. Applying Theorem 2 gives

$$\begin{aligned} B_s c &= \xi_0 - \Gamma_h r_f, & u_h(0) &= r_f + Q_s c, \\ u_h(t) &= (\mathcal{K} f_h)(t) + Q_s e^{T_s t} c. \end{aligned} \quad (4.28)$$

The Green operator $\mathcal{K}$ is (3.7) for the discrete equilibrium generator. Its forced path has initial profile r_f and already satisfies admissibility at infinity. The stable adjustment changes that profile to match the inherited distribution. In this experiment $\xi_0 = 0$, so $B_s c = -\Gamma_h r_f$ cancels the forced path's initial distribution component. Values can adjust at the announcement. Thereafter c remains fixed, while $e^{T_s t}$ propagates its contribution and $\mathcal{K} f_h$ incorporates the productivity forcing. Thus the initial boundary solve selects a complete convergent transition, not just an impact response. Appendix C.7 gives the equivalent particular-solution formula used by the Julia implementation. It evaluates the path using exponentials of the stable block.

To obtain the plotted aggregate IRFs, recover the cell-mass perturbation $\delta m_h(t) = Z_m \xi(t)$, and use the policy derivatives from Step 2:

$$\begin{aligned} \delta K(t) &= k_h^\top \xi(t), & \delta p(t) &= P_K \delta K(t) + P_Z \delta Z(t), \\ \delta c_h(t) &= C_c v_h(t) + D_c \delta p(t), & \delta C_{\text{agg}}(t) &= \bar{m}_h^\top \delta c_h(t) + \bar{c}_h^\top Z_m \xi(t). \end{aligned}$$

Here C_c, D_c denote the grid policy derivatives, $\bar{m}_h$ is the stationary cell-probability vector, and $\bar{c}_h$ is stationary household consumption. Aggregate consumption changes both because households change their policies and because population mass moves across states with different consumption levels. The first component of δp is the interest-rate response. Figure 2 plots $100\delta K/\bar{K}$, $100\delta C_{\text{agg}}/\bar{C}_{\text{agg}}$, and $10^4\delta r$, where $\bar{C}_{\text{agg}} = \bar{m}_h^\top \bar{c}_h$. Inherited capital cannot jump, while consumption rises at the announcement. Subsequent saving builds capital and eventually lowers its return below steady state.

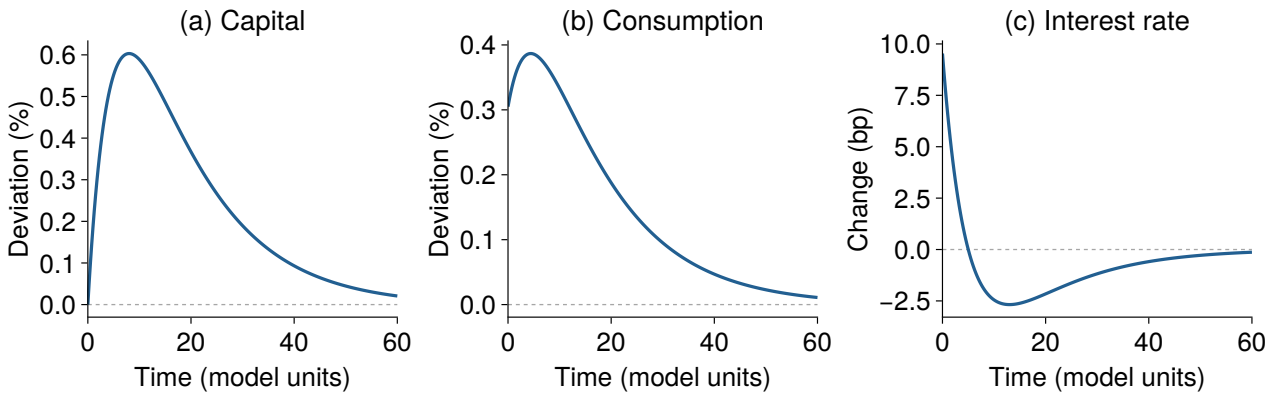


Figure 2: Impulse responses to a one-percent initial productivity increase

Notes: $N_a = 100$, $\delta Z(t) = 0.001e^{-0.2t}$, and $\xi_0 = 0$. Capital and consumption are percentage deviations from their steady-state levels. Interest is a basis-point change. The curves are the continuous-time linear responses in (4.28), sampled every 0.25 model-time units. The displayed endpoint is a plotting window, not a terminal condition. Admissibility is imposed on the infinite horizon.

The independent Schur benchmark agrees with the computed spectral projection to 1.76×10^{-9} in relative Frobenius norm on the larger grid. Appendix C.7 reports the remaining verification checks. Section 4.6 uses this same continuous-time response as the reference for a finite-prefix SSJ calculation and explains how the projection supplies its exact continuation.

4.6 Connection to finite-horizon SSJ

The projected IRF in Section 4.5 is a continuous-time response of the fixed DEC spatial economy. A finite-prefix SSJ calculation can recover the same equilibrium by using the stable split to account for the future beyond its matching date. The two continuous-time formulations are exactly equivalent. Time discretization makes the implemented SSJ an approximation to that common response. The matching date is a computational interface, distinct from the announcement date and the endpoint displayed in an IRF figure.

The same admissibility condition at a later date. Suppose the reduced equilibrium dynamics beyond T are

$$\dot{u} = A_\infty u + f(t), \quad t \geq T, \quad (4.29)$$

with the dichotomy of Assumption 2. The stable trace can adjust to the distribution and other states inherited at T , whereas future forcing determines the unstable trace:

$$h_u(T) = - \int_T^\infty e^{A_\infty(T-r)} P_u f(r) dr. \quad (4.30)$$

The exact continuation condition is

$$\boxed{P_u u(T) = h_u(T)}. \quad (4.31)$$

It is imposed in $E_u = \text{Ran } P_u$, without counting redundant coordinate rows as additional restrictions. With no remaining forcing, it requires $u(T) \in E_s$. With future shocks, it selects a parallel affine set. This is the date- T version of the saddle-path selection used at date zero, including the generalized discounting of future forcing.

Theorem 3 (Tail-exact equivalence). *Assume the stationary tail satisfies Assumption 2 and $f \in L^2([T, \infty); E)$. Restriction to $[0, T]$ gives a bijection between admissible half-line solutions and prefix solutions satisfying the original initial conditions and (4.31). Each compatible terminal trace has a unique admissible continuation. Consequently, exact prefix solution and exact continuation introduce no horizon truncation error, independently of the matching date within the stationary region. If the complete half-line problem has a unique admissible response, every such formulation recovers that response.*

Appendix D.5 gives the continuation formula and proof. The theorem preserves the original existence and uniqueness question. It does not make an incompatible shock admissible. It also permits time-varying coefficients before T . Freezing coefficients that only approach a stationary limit instead introduces a tail approximation, as discussed in Appendix D.11.

How the condition enters an SSJ calculation. At T , the inherited distribution generally remains away from steady state. Its subsequent adjustment affects equilibrium prices, and households anticipate those prices when making decisions before T . The terminal relation therefore enters each household response used to evaluate the aggregate market residual and its Jacobian. It corrects both the feedback from trial prefix prices and the forcing from announced future shocks. Appending a tail after solving a zero-terminated SSJ would leave the earlier decisions distorted. When internal elimination is regular, aggregate price or quantity paths remain the SSJ unknowns. Appendix D.8 gives the terminal-feedback solve and Algorithm 2.

For the Aiyagari productivity experiment, the same projection and forced trace used to construct Figure 2 give

$$P_{u,h} \begin{pmatrix} v_h(T) \\ \xi(T) \end{pmatrix} = r_f e^{-\nu T}. \quad (4.32)$$

The scalar exponential scales the forced unstable trace. The stable part of the response is still selected by the prefix equations and inherited distribution. Since the boundary test in Section 4.5 passes, this condition also admits the graph $v_h(T) = Q\xi(T) + q_Z\delta Z(T)$, derived in Appendix D.7. Substituting it into the household terminal-response map from Step 3 completes the aggregate capital equation from Step 4.

Time refinement and matching-date comparisons. The numerical SSJ represents prices as constant on each prefix cell and collocates firm pricing at the cell midpoint. Household integration within each cell is exact, but the representation of the equilibrium price path still introduces time-discretization error. Table 3 compares the computed capital response with the continuous-time projected reference on the same 100-node asset grid.

Table 3: Capital-path accuracy of the aggregate SSJ calculation

Matching date T	Time step Δt	With continuation	With $v_h(T) = 0$
20	1.00	0.006492	0.141352
20	0.50	0.001699	0.146257
20	0.25	0.000435	0.148659
40	0.50	0.001699	0.035810

Notes: $N_a = 100$. Errors are the maximum absolute difference at prefix cell midpoints from the continuous-time projected response, in percentage points of steady-state capital. The two calculations use the same time grid. The continuation condition retains both endogenous future adjustment and the remaining productivity shock.

At $T = 20$, halving the time step roughly quarters the error with the continuation condition. These are numerical refinement results, rather than a general convergence-rate theorem. Convergence to the fixed-grid reference requires a consistent, stable prefix discretization, with projection and solve errors tending to zero. Convergence to a continuous-state economy additionally requires spatial and differentiated consistency and appropriate spectral stability. By contrast, imposing $v_h(T) = 0$ at a fixed T changes the boundary-value problem: reducing the time step alone does not remove that error.

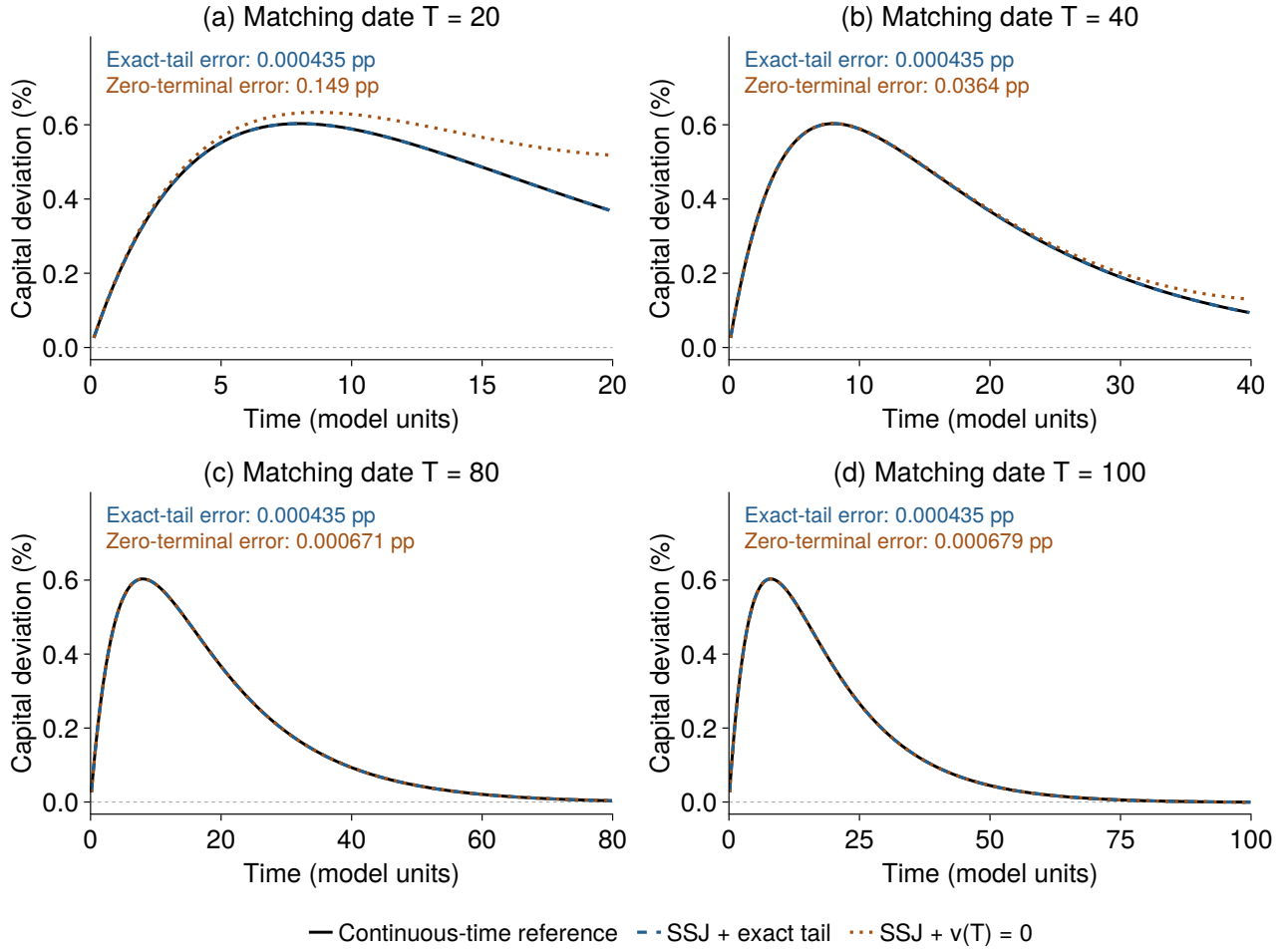


Figure 3: Capital responses across matching dates

Notes: The calibration and productivity shock are those of Figure 2. Each panel compares the continuous-time projected reference with the aggregate SSJ using exact continuation and with the SSJ imposing $v_h(T) = 0$, for $T = 20, 40, 80, 100$. All calculations use $N_a = 100$ and $\Delta t = 0.25$; the reference is sampled at the same cell midpoints. The endpoint of each panel is its matching date. Capital responses are percentage deviations from steady state. Numerical errors remain even where the continuation and reference curves are visually indistinguishable. Panel error labels report the maximum absolute discrepancy from the reference at prefix cell midpoints, in percentage points.

Figure 3 varies the matching date while holding the time step fixed. The exact continuation retains both the endogenous future adjustment and the remaining productivity shock at every date. Moving the artificial zero terminal value farther away reduces its boundary distortion, but its accuracy depends on the horizon. At $T = 80$ and 100 , the remaining total discrepancies are of the same order as the prefix discretization error. That total need not decrease monotonically with T . In the matched formulation, the continuation itself introduces no such horizon error. Remaining differences reflect the numerical prefix and tail evaluations. Appendix C.7 supplies implementation checks and reproduction commands, and Appendix D.10 isolates the same mechanism in an exactly solvable example.

4.7 Nominal feedback and multiple heterogeneous blocks

To see how a familiar policy criterion fits the same geometry, consider a continuous-time New Keynesian subsystem

$$\dot{y} = \sigma^{-1}(\phi_\pi - 1)\pi + f_y, \quad \dot{\pi} = \rho\pi - \kappa y + f_\pi, \quad \rho, \sigma, \kappa > 0. \quad (4.33)$$

Here y and π denote the output-gap and inflation deviations, respectively. There are no predetermined endogenous variables in this two-variable illustration. The dynamics matrix has trace ρ and determinant $\kappa(\phi_\pi - 1)/\sigma$. For $\phi_\pi > 1$, both roots have positive real part: $E_s = \{0\}$, matching the zero-dimensional boundary space, and a unique admissible response exists for L^2 forcing. For $\phi_\pi < 1$, there is one stable direction and therefore a one-dimensional homogeneous solution family. At equality a center root violates the dichotomy assumption.

A HANK extension appends household distribution states, fiscal dynamics, and additional aggregate restrictions. The criterion remains the rank of the stable-trace map of the complete system. The scalar inequality in (4.33) is not asserted to characterize all such extensions. With household and firm heterogeneity, one stacks both mass-zero distribution spaces and both optimality blocks. The fiber product in (2.10) and the Schur argument remain valid whenever their regularity assumptions hold.

5 Conclusion

The household policy response, the distribution response, and the general-equilibrium price response form one economic feedback system. Its continuous-time sequence-space representation is a boundary-value relation linking optimal decisions, mass-preserving transport, and market clearing. Its geometry separates three requirements: admissible dynamics, compatible shocks, and a unique intersection with predetermined initial data. Under closed-range and dichotomy assumptions, these requirements reduce to kernel and cokernel conditions on a stable-trace map. The same construction produces an exact infinite-tail boundary condition.

This connection provides a precise route to eliminating time-horizon truncation error. A finite prefix with its exact admissible continuation represents the half-line equilibrium, including future forcing. Analytical fake-news household factors also construct the finite-grid spectral

split through a frequency-dependent price solve. Their dated responses evaluate the reference household Jacobian. A terminal feedback solve supplies the continuation-value correction within each Jacobian action. The calibrated Aiyagari calculation uses the direct projected IRF as a continuous-time reference on the same spatial grid. Its time-refinement and matching-date comparisons illustrate the distinction between numerical approximation and an artificial terminal restriction. Valid elimination of internal paths retains aggregate sequences as the equilibrium unknowns and preserves the same closure defects. DEC provides a compatible realization of the transport component, while analytic policy responses and matrix-free methods supply possible computational gains.

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